The Reverse Mathematics of CAC for trees

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Reverse mathematics

- Second order arithmetics
- Order classical theorems of arithmetics by power
- Need a weak arithmetic to compare them:
  - T_1 is weaker than T_2 is $\mathsf{RCA}_0 \vdash T_2 \Rightarrow T_1$ and $\mathsf{RCA}_0 \nvdash T_1 \Rightarrow T_2$
- Big five
  - $\mathsf{RCA}_0 = \mathsf{Q} + I\Sigma^0_1 + C\Delta^0_1$, "constructive mathematics"
  - $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \mathsf{WKL}$
  - $\mathsf{ACA}_0 = \mathsf{RCA}_0 + C\Sigma^0_1$ ($\Leftrightarrow$ comprehension over all first order arithmetic formula)
  - $\mathsf{ATR}_0 = \mathsf{ACA}_0 + TF$ (TF is transfinite constructions)
  - $\Pi^1_1 - \mathsf{CA}_0 = \mathsf{RCA}_0 + C\Pi^1_1$
- For $\vdash$ uses direct constructive proofs
- For $\nvdash$, one can use computability arguments

Turing ideal

Definition

A model $\mathcal{M}$ of second order arithmetic is an ω-structure if its first order elements are standard

Definition (Turing ideal)

The set $\mathcal{I}$ is a *Turing ideal* if it is closed by Turing reduction and join :

- $\forall X \in \mathcal{I}, \forall Y, Y \leqslant_T X \Rightarrow Y \in \mathcal{I}$
- $\forall X, Y \in \mathcal{I}, X \oplus Y \in \mathcal{I}$ where $X \oplus Y = 2X \cup (2Y + 1)$

Proposition (Friedman)

An ω-model is a model of RCA_0 if and only if its second order part is a Turing ideal

Reductions

- We consider statements $P = \forall X.(I(X) \Rightarrow \exists Y.Q(X,Y))$ where I and Q are first order formulas
- It makes P a *problem*: for all set X such that $I(X)$ (the *instance*), any set Y such that $Q(X,Y)$ is a *solution* to the instance X of P

Definition

A Turing ideal $\mathcal{J}$ satisfies a problem P, denoted by $\mathcal{J} \models P$ if all instance $X \in \mathcal{J}$ of P has a solution in $\mathcal{J}$

Definition (ω-reduction)

A problem P is ω-reducible to a problem Q, denoted by $P \leqslant_\omega Q$, if for all Turing ideal $\mathcal{J}$, $\mathcal{J} \models Q \implies \mathcal{J} \models P$

Proving $\mathsf{RCA}_0 + Q \nvdash P$

Call an ω-model a theory of model which is an ω-structure. A corollary of Friedman's result is

Claim

$P \leqslant_\omega Q$ if and only if any ω-model of $\mathsf{RCA}_0 + Q$ is also a model of $\mathsf{RCA}_0 + P$

Claim

$P \nleqslant_\omega Q$ implies $\mathsf{RCA}_0 + Q \nvdash P$

We have a tool to prove that a statement is weaker than another: find a Turing ideal $\mathcal{J}$ which satisfies Q but not P

CAC

For a (partial) order $\langle E, \prec \rangle$, a *chain* is a set X such that $\langle X, \prec \rangle$ is total; an *antichain* is a set X such that $\forall x, y \in X$, $x \perp y$ (meaning $x \nprec y \wedge y \nprec x$)

Statement (CAC – Chain/Antichain theorem)

All infinite partial order has either an infinite chain or an infinite antichain

Statement (RT^2_2 – Ramsey theorem pour pairs and two colors)

All coloring of pairs of integers $c : [\mathbb{N}]^2 \to 2$ has a monochromatic $X \subseteq \mathbb{N}$ that is $\exists i \in 2, \forall x, y \in X, c(\{x, y\}) = i$

Theorem (Cholak, Jockusch and Slaman)

$\mathsf{RCA}_0 \nvdash \mathsf{CAC} \Rightarrow \mathsf{RT}^2_2$

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \Rightarrow \mathsf{CAC}$: define a coloring such that $\{x, y\}$ has color 1 if its elements are comparable, and 0 otherwise

CAC for trees

A (binary) tree is a subset of $\mathbb{N}^{<\omega}$ ($2^{<\omega}$) closed by prefix.

Statement (CAC for (c.e.) (binary) trees, Binns et al.)

Every (c.e.) (binary) infinite tree has an infinite path or an infinite antichain.

Computably enumerable means that the set is not in the model but can be approximated by elements in the model

Theorem (Binns et al.)

$\mathsf{RCA}_0 + \mathsf{WKL} \nvdash$ CAC for binary trees

We will see that this statement is robust w.r.t reverse mathematics and is equivalent to several problems

Definition (Completely branching tree)

A node σ of a tree is a *split node* when there is $n_0, n_1 \in \mathbb{N}$ such that $\sigma n_0 \in T \wedge \sigma n_1 \in T$. A tree is *completely branching* if all its nodes are either a split node or a leaf.

The following statement was introduced by Conidis, motivated by the reverse mathematics of commutative nœtherian rings.

Definition (TAC, Conidis, tree antichain theorem)

Any infinite c.e. binary tree which is completely branching, contains an infinite antichain.

CAC for trees equivalent statements

Theorem

The following are equivalent over RCA_0:

1. CAC for trees
2. CAC for c.e. trees
3. CAC for binary c.e. trees
4. $\mathsf{TAC} + B\Sigma^0_2$

TAC and WKL

Theorem

For any low set P, there exists a computable instance of TAC with no P-computable solution.

Corollary

$\mathsf{RCA}_0 + \mathsf{WKL} \nvdash \mathsf{TAC}$

since there exists a model of $\mathsf{RCA}_0 + \mathsf{WKL}$ below a low set.

Using measure

Proposition

The measure of the oracles computing a solution for a computable instance of TAC is 1.

COH states that for sets A_n there is a set U almost included in A_n or $\mathbb{N} \setminus A_n$ for all n.

Corollary

$\mathsf{RCA}_0 + \mathsf{TAC} \nvdash \mathsf{COH}$.

COH has a computable instance such that the measure of the oracles computing a solution is 0 (Astor et al).

Proposition

$\mathsf{RCA}_0 + \mathsf{TAC} \nvdash B\Sigma^0_2$ *and* $\mathsf{RCA}_0 + \mathsf{TAC} \nvdash$ CAC for trees

from a result from Slaman about a combinatorial statement named 2RAN we proved stronger than TAC and which dœs not implies $B\Sigma^0_2$ over RCA_0

ADS and EM

RT^2_2 admits a famous decomposition over RCA_0: into the Ascending Descending Sequence theorem (ADS) and the Erdös-Moser theorem (EM).

Disjunctive part

Statement (ADS)

All infinite linear order admits an infinite increasing sequence or an infinite decreasing sequence

Compacity part

Statement (EM- Erdös–Moser)

A tournament is an irreflexive binary relation such that for all $x \neq y$, either $x\mathcal{R}y$ or $y\mathcal{R}x$. Every infinite tournament T has an infinite transitive subtournament.

ADS and EM

Theorem (Lerman, Solomon and Towsner + Hirschfeldt and Shore)

$\mathsf{RCA}_0 \vdash \mathsf{EM} + \mathsf{ADS} \Rightarrow \mathsf{RT}^2_2$ *but* $\mathsf{RCA}_0 \nvdash \mathsf{EM} \Rightarrow \mathsf{RT}^2_2$ *and* $\mathsf{RCA}_0 \nvdash \mathsf{ADS} \Rightarrow \mathsf{RT}^2_2$

A tournament can be seen as a coloring: for $x < y$, $x\mathcal{R}y$ means $c(\{x,y\}) = 1$ and $y\mathcal{R}x$ means $c(\{x,y\}) = 0$

Coloring —EM $\rightarrow$ transitive coloring —ADS $\rightarrow$ homogeneous set.

Proposition

$\mathsf{RCA}_0 \vdash \mathsf{ADS} \Rightarrow$ CAC for trees *and* $\mathsf{RCA}_0 \vdash \mathsf{EM} \Rightarrow$ CAC for trees

Statements with forbidden patterns

Several statement (EM, RT^2_2, ADS) follow the same pattern: for some coloring with one type of restriction, one can find an infinite set which makes the coloring of another type of restriction.

Here restriction = some set of forbidden patterns. This allows to produce new statements.

Definition (Semi-heredity)

A coloring $f : [\mathbb{N}]^2 \to 2$ is semi-hereditary for the color $i < 2$ if $\forall x < y < z, f(x, z) = f(y, z) = i \implies f(x, y) = i$.

Statement (SHER Dorais et al.)

For any semi-hereditary coloring, there exists an infinite homogeneous set.

Theorem

SHER *and* CAC for trees *are equivalent over* RCA_0.

Stable variants

Definition

A coloring $f : [\mathbb{N}]^2 \to k$ is **stable** if for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists. A linear order $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}})$ is **stable** if it is of order type $\omega + \omega^*$.

A tree $T \subseteq \mathbb{N}^{<\omega}$ is **stable** when for every $\sigma \in T$ either $\forall^\infty \tau \in T, \sigma \perp \tau$ or $\forall^\infty \tau \in T, \sigma \not\perp \tau$

Proposition

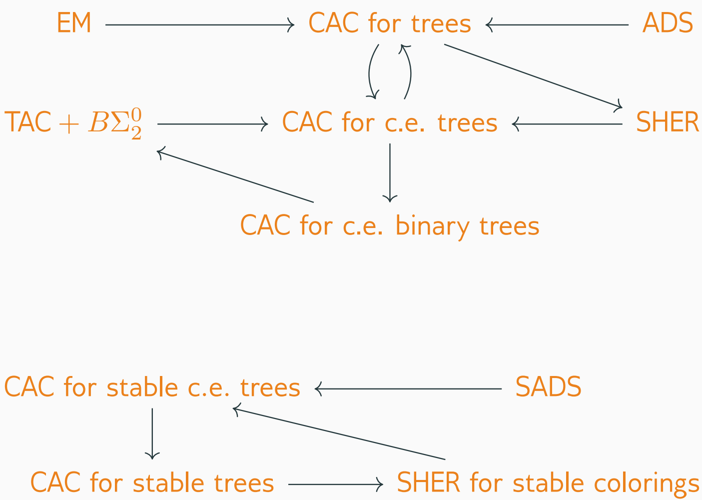
$\mathsf{RCA}_0 \vdash \mathsf{SADS} \implies$ CAC for stable c.e. trees

Corollary

The following are equivalent over RCA_0*:*

1. CAC for stable trees
2. CAC for stable c.e. trees
3. SHER *for stable colorings*

Summary


EM
CAC for trees
ADS
$\mathsf{TAC} + B\Sigma^0_2$
CAC for c.e. trees
SHER
CAC for c.e. binary trees
CAC for stable c.e. trees
SADS
CAC for stable trees
SHER for stable colorings

Open questions

Question

What is the first-order part of TAC?

Question

Dœs every computable instance of CAC for trees admit a low solution?

Question

Is there some X such that for every computable instance T of CAC for trees, every DNC function relative to X computes a solution to T?