On end extensions of models of open induction

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Theorem 1 (MacDowell-Specker, 1961). Every model of PA has a proper elementary end extension.

Aim. Miniaturize the MacDowell-Specker theorem

$I\Sigma_n$: induction for Σ_n formulas (plus base theory)

$B\Sigma_n$: $I\Delta_0$ + collection for Σ_n formulas

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Theorem 2 (Paris & Kirby 1978).
(a) For all $n \in \mathbb{N}$, $I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n$
and the converse implications are false.
(b) For $n \geq 2$, if M is a countable model of $B\Sigma_n$, then M has a proper Σ_n-elementary end extension K satisfying $I\Delta_0$.

Problem 1. For $n \geq 2$, does every model of $B\Sigma_n$ have a proper Σ_n-elementary end extension satisfying $I\Delta_0$?

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Theorem 4 (Wilkie & Paris 1989). If M is a countable model of $B\Sigma_1 + exp$, then there exists K such that $M \subset_e K$ and K satisfies $I\Delta_0$. (*exp* expresses "exponentiation is total")

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Lemma 6. If M, K satisfy PA and K is strongly definable in M, then M is isomorphic to an initial segment of K.

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C. Dimitracopoulos and V. Paschalis (2016 & 2020). Alternative proofs of Theorems 3 and 7, using variants of ACT. The main ideas for the proofs are:

- using induction **in the metalanguage**, construct a consistent theory T in an extension of LA (in the given model), via a lemma on the possibility of witnessing bounded existential quantifiers with appropriate constants and
- take as universe of the required extension an appropriate set of elements definable in a model of T.

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Remark 2. A positive solution to Problem 5(b), would
(i) combined with Fact 2, imply ($\Leftarrow$) of Theorem 9
(ii) give a positive solution to Problem 5(a), thus generalizing Theorem 4 (Wilkie & Paris 1989).