(short) bounded recursions and Δ_0-definability

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An open problem

$$\Delta_0^{\mathbb{N}} \subseteq \mathcal{E}_\star^0$$

Equality or not ?

An open problem in other terms

Let us suppose that

$$\begin{cases} f(\vec{u},0) = u_0 \\ f(\vec{u},i+1) = h(\vec{u},i,f(i)) \end{cases}$$

and

$$\begin{cases} f(\vec{u},y) \leq Max\{\vec{u},y\} \\ Z = h(\vec{u},i,z) \text{ is } \Delta_0 - \text{definable} \end{cases}$$

Is $Z = f(\vec{u},y)$ Δ_0-definable ?

Plan

- Basic informations

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- Short bounded recursions : known results

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- Main results and ideas of proofs

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- Short bounded recursions : known results
- Main results and ideas of proofs
- Conclusion

What is Δ_0−definability ?

x is not prime nor 0 nor 1

$$(\exists u) \qquad (\exists v) \qquad (x = uv) \wedge (u \neq x) \wedge (v \neq x)$$

What is Δ_0−definability ?

x is not prime nor 0 nor 1

$$(\exists u)_{u<x}\,(\exists v)_{v<x}\,(x = uv) \wedge (u \neq x) \wedge (v \neq x)$$

What is Δ_0–definability ?

Exemple

$$z = x^y$$

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$y = 2 \wedge z = x.x$ and $y = 3 \wedge z = x.x.x$ and ...

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$$f(x,0) = x \qquad f(x,i+1) = f(x) \times x$$

What is Δ_0-definability ?

Exemple

$$z = x^y$$

$y = 2 \wedge z = x.x$ and $y = 3 \wedge z = x.x.x$ and ...

$$f(x,0) = x \qquad f(x,i+1) = f(x) \times x$$

IS Δ_0-definable

What is Δ_0−definability ?

The graph of the following function

$$\begin{cases} f(0) = 0 \\ f(i+1) = (f(i)+1) \mod 2 & \text{if } i \text{ is prime} \\ f(i+1) = f(i) & \text{if } i \text{ is not prime} \end{cases}$$

IS NOT KNOWN TO BE Δ_0−definable

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IS NOT KNOWN TO BE Δ_0−definable

****

BUT the graph of $f(lh_2(x))$ IS Δ_0−definable

What is Δ_0–definability ?

$$\begin{cases} f(0) = 0 & \\ f(i+1) = \quad (f(i)+1) \mod 2 & \text{if i is prime} \\ f(i+1) = \quad f(i) & \text{if i is not prime} \end{cases}$$

$z = f(y)$ iff

What is Δ_0–definability ?

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$z = f(y)$ iff

$\exists(z_0, z_1, ..., z_y) \in \{0,1\}^{y+1}$

$$\begin{cases} z_0 = 0 \\ \forall i \leq y-1 \quad \begin{cases} z_{i+1} = & (z_i + 1) \mod 2 & \text{if } i \text{ is prime} \\ z_{i+1} = & z_i & \text{if } i \text{ is not} \end{cases} \\ z = z_y \end{cases}$$

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$z = f(y)$ iff

$\exists (z_0, z_1, ..., z_y) \in \{0,1\}^{y+1}$ $\exists Z \leq 2^{y+1}$

$$\begin{cases} z_0 = 0 \\ \forall i \leq y-1 \quad \begin{cases} z_{i+1} = & (z_i + 1) \mod 2 & \text{if } i \text{ is prime} \\ z_{i+1} = & z_i & \text{if } i \text{ is not} \end{cases} \\ z = z_y \end{cases}$$

Short recursions, long recursions

$$\begin{cases} \bar{f}(\vec{u}, 0) = u_0 \\ \bar{f}(\vec{u}, i+1) = h(\vec{u}, i, \bar{f}(\vec{u}, i)) \end{cases}$$

long recursions	short recursions
$\bar{f}(\vec{u}, y)$	$f(\vec{u}, y) = \bar{f}(\vec{u}, lh_2(y))$

Recursions and Δ_0–definability

transition function		long rec	short rec
$z + 1$ if $R(\vec{u}, i)$, else z	R is Δ_0	$\Delta_0^\sharp$	Δ_0 [1]
az	a is a variable	Δ_0 [2]	
$z + b(\vec{u}, i)$	$b(\vec{u}, i) \leq polyn(\vec{u})$ $Graph(b)$ is Δ_0	$\Delta_0^\sharp$ [3]	Δ_0 [4]
$a(\vec{u}, i) \times z$	$Graph(a)$ is Δ_0	Δ_0 [5]	
$(a \times z) \bmod m$	a, m are variables	Δ_0 [6]	

Recursions and Δ_0-definability

Sequences issued from Euclid's algorithm

$f(a, b, 0) = a$ $\qquad$ $f(a, b, 1) = b$

$$f(a, b, i+2) = f(a, b, i) \bmod f(a, b, i+1)$$

It is essentially a short recursion and the graph of f is Δ_0-definable [7]

Recursions and Δ_0-definability

Linear recurrence sequences

$$L(\vec{x}, i+k) = \sum_{j=0}^{k-1} a_j \times L(\vec{x}, i+j)$$

(k is a constant).

The graph of L is Δ_0-definable [8]

Main results and ideas of proofs

Result I

The short recursion with transition function

$$H(a, c, b, d, x, z, i) = \begin{cases} az + b \text{ if } (x)_i = 1 \\ cz + d \text{ if } (x)_i = 0 \end{cases}$$

defines a function with a Δ_0- definable graph.

$(\bar{x})_i$ the i-th binary digit of x.

Result I : some ideas of the proof

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$\bar{x} \in \{0,1\}^\star$ is the binary expansion of x

$$\bar{x} = 0^{\alpha(x,0)}1^{\beta(x,0)}0^{\alpha(x,1)} \ldots 1^{\beta(x,lh_2(x)-2)}0^{\alpha(x,lh_2(x)-1)}1^{\beta(x,lh_2(x)-1)}$$

$$L(x,i) = \sum_{j=0}^{j=i-1} \alpha(x,j) + \beta(x,j)$$

$$L_0(x,i) = \sum_{j=0}^{j=i-1} \alpha(x,j) \qquad L_1(x,i) = \sum_{j=0}^{j=i-1} \beta(x,j)$$

Result I : some ideas of the proof

$$\bar{F}(a, c, b, d, x, L(i)) =$$

$$a^{L_1(x,i)} c^{L_0(x,i)} \bar{F}(a, c, b, d, x, 0)$$

$$+ \frac{d}{c-1} \left(\sum_{j=0}^{j=i} a^{L_1(x,i) - L_1(x,j)} c^{L_0(x,i) - L_0(x,j+1)} (c^{\alpha(x,j)} - 1) \right)$$

$$+ \frac{b}{a-1} \left(\sum_{j=0}^{j=i} a^{L_1(x,i) - L_1(x,j)} c^{L_0(x,i) - L_0(x,j)} (a^{\beta(x,j)} - 1) \right)$$

Result I : some ideas of the proof

And similar formulas for

$$L(x,i) \leq y < L(x,i) + \alpha(x,i+1)$$

and

$$L(x,i) + \alpha(i+1) \leq y < L(x,i+1)$$

Result I : some ideas of the proof

$z = L(x, i)$ is equivalent to

$$(\bar{x})_z = 1 \wedge (\bar{x})_{z-1} = 0 \wedge i = card\{j < i; (\bar{x})_i = 1 \wedge (\bar{x})_{i+1} = 0\}$$

with $i \leq lh_2(x)$

$$\alpha(x, i) + \beta(x, i) = L(x, i + 1) - L(x, i)$$

$z = \beta(x, i)$ is equivalent to without paying attention to borders !

$$\exists u\,((u = L(x, i + 1) + 1) \wedge z = card\{j < u; (\bar{x})_i = 1 \wedge (\bar{x})_{i-1} = 1\})$$

Result I : some ideas of the proof

$$\bar{F}(a,c,b,d,x,L(i)) =$$

$$a^{L_1(x,i)}c^{L_0(x,i)}\bar{F}(a,c,b,d,x,0)$$

$$+\frac{d}{c-1}\left(\sum_{j=0}^{j=i} a^{L_1(x,i)-L_1(x,j)}c^{L_0(x,i)-L_0(x,j+1)}(c^{\alpha(x,j)}-1)\right)$$

$$+\frac{b}{a-1}\left(\sum_{j=0}^{j=i} a^{L_1(x,i)-L_1(x,j)}c^{L_0(x,i)-L_0(x,j)}(a^{\beta(x,j)}-1)\right)$$

Result I : some ideas of the proof

The main step for studying the case where a, b, c, d are variables :

Lemma. the following relation is Δ_0-definable

$$\left(Z = \sum_{j=0}^{j=i-1} \gamma(x,j) \right) \wedge (i \leq lh_2(y))$$

where

$\star\ \forall j \leq i\, (\gamma(x,j) \leq b(x,y))$

$\star$ $\log_2(b(x,y))$ is a polylog. of the variables

$\star$ the graph of γ and b are Δ_0-definable

Result I : some ideas of the proof

$$\left(Z = \sum_{j=0}^{j=i-1} \gamma(x,j)\right) \wedge (i \leq lh_2(y))$$

is equivalent to :

$(i \leq lh_2(y)) \wedge Z \leq b(x,y) \times lh_2(y)$ and

$$\forall p \leq 2\log_2\left(b(x,y) \times lh_2(y)\right), p \text{ prime}$$

$$\left(Z \equiv \sum_{j=0}^{j=i-1} \gamma(x,j)\right) \mod p$$

Result I : some ideas of the proof

now

$$\left(\sum_{j=0}^{j=i-1} \gamma(x,j)\right) \mod p$$

is equal to

$$\left(\sum_{k=0}^{k=p-1} k \times Card\{j \leq i-1; \gamma(x,j) \equiv k \mod p\}\right) \mod p$$

Result II

The short recursion with transition function

$$h_{a_1,a_2}(m_1, m_2, z) = (a_2 \, (a_1 z \bmod m_1) \, \bmod m_2)$$

defines a function with a Δ_0- definable graph.

Result II : some ideas of the proof

Result II : some ideas of the proof

$z = \bar{f}_{a_1,a_2}(m_1, m_2, u, y)$ and $0 \leq u \leq m_2 - 1$

is equivalent to
$\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ exists such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq x \leq m_2 - 1) \wedge (\mathbf{z}_0 = u \wedge (\mathbf{z}_y = z)$$

$$\forall i \leq y - 1 \ \mathbf{z}_{j+1} = h_{a_1,a_2}(m_1, m_2, \mathbf{z}_j)$$

Result II : some ideas of the proof

$z' = h_{a_1,a_2}(m_1, m_2, z)$ and $0 \leq z \leq m_2 - 1$

is equivalent to

$0 \leq z \leq m_2 - 1$ and $k_1 \leq m_2 - 1$ and $k_2 \leq a_2 - 1$ exist such that

$$\begin{cases} 0 \leq z' \leq m_2 - 1 \\ z' + k_2 m_2 \leq a_2(m_1 - 1) \\ a_1 a_2 z - z' = a_2 k_1 m_1 + k_2 m_2 \end{cases}$$

Result II : some ideas of the proof

$z = \overline{f}_{a_1,a_2}(m_1, m_2, u, y)$ and $0 \leq x \leq m_2 - 1$

is equivalent to

Result II : some ideas of the proof

Exist $\mathbf{k_1} \in \{0, 1, ..., m_2 - 1\}^y$ and $\mathbf{k_2} \in \{0, 1, ..., a_2 - 1\}^y$ and $\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq x \leq m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z)$$

$$\forall i \leq y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \leq (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

where $S_{\mathbf{k}}(i) = \sum_{j=0}^{j=i} \mathbf{k}_{i-j}(a_1 a_2)^j$

Result II : some ideas of the proof

Exist $\mathbf{k_1} \in \{0, 1, ..., m_2 - 1\}^y$ and $\exists K_2 \leq x_0^\gamma$ and $\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ such that

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq x \leq m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z)$$

$$\forall i \leq y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \leq (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

where $S_{\mathbf{k}}(i) = \sum_{j=0}^{j=i} \mathbf{k}_{i-j}(a_1 a_2)^j$

Result II : some ideas of the proof

An easy remark :

If $m_2 \geq 1 + a_2(m_1 - 1)$ then

$$h_{a_1,a_2}(m_1, m_2, z) = a_2\,(a_1 z \bmod m_1)$$

Δ_0-definability from Hesse theorem, even for

$$h(a_1, a_2, m, z) = a_1\,(a_2 z \bmod m)$$

Result II : some ideas of the proof

Exist $\mathbf{k_1} \in \{0, 1, ..., m_2 - 1\}^y$ and $\exists K_2 \le x_0^\gamma$ and $\mathbf{z} \in \{0, 1, ..., m_2 - 1\}^{y+1}$ such that

$$(0 \le z \le m_2 - 1) \wedge (0 \le x \le m_2 - 1) \wedge (\mathbf{z}_0 = u) \wedge (\mathbf{z}_y = z)$$

$$\forall i \le y$$

$$\begin{cases} \mathbf{z}_i + S_{\mathbf{k_2}}(i-2)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 \le (a_1 a_2)^i (m_1 - 1) \\ \mathbf{z}_i + S_{\mathbf{k_2}}(i-1)m_2 + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x \end{cases}$$

where $S_{\mathbf{k}}(i) = \sum_{j=0}^{j=i} \mathbf{k_2}_{i-j}(a_1 a_2)^j$

Result II : some ideas of the proof

$$(0 \leq z \leq m_2 - 1) \wedge$$

$$\forall i \leq y$$

$$\begin{cases} \\ \mathbf{z}_i + a_2 S_{\mathbf{k_1}}(i-1)m_1 = (a_1 a_2)^i x - S_{\mathbf{k_2}}(i-1)m_2 \end{cases}$$

with $m_2 \leq a_2(m_1 - 1)$

Result II : some ideas of the proof

$$\exists K_2 \leq x_0^{\gamma}$$

$$(0 \leq z \leq m_2 - 1) \wedge (0 \leq x \leq m_2 - 1)$$

$$\forall i \leq y \quad \exists \zeta \leq m_2 - 1 \exists \chi \leq (m_2 - 1)^{\frac{(a_1 a_2)^{i+1} - 1}{a_1 a_2 - 1}}$$

$$\begin{cases} \zeta + \chi m_2 + a_2 S_{\mathbf{k_2}}(i-1) m_1 \leq (a_1 a_2)^i (m_1 - 1) \\ \zeta + \chi m_2 + a_2 S_{\mathbf{k_2}}(i-1) m_1 = (a_1 a_2)^i x \end{cases}$$

where $S_{\mathbf{k_2}}(i) = \sum_{j=0}^{j=i} \mathbf{k_2}_{i-j} (a_1 a_2)^j = \left\lfloor \frac{K}{(a_1 a_2)^{i+1}} \right\rfloor$

and $\zeta = ((a_1 a_2)^i x - m_2 S_{\mathbf{k_2}}(i-1)) \mod (a_2 m_1)$

and $\chi = \left\lfloor \frac{(a_1 a_2)^i x - m_2 S_{\mathbf{k_2}}(i-1)}{a_2 m_1} \right\rfloor$

Some generalizations and conclusion

Some generalizations and conclusion

Consequence 1. The short recursion for transition function

$$h_{R,(a,c),(b,d)}(\vec{u}, z, i) = \begin{cases} az + b \text{ if } R(\vec{u}, i, z) \\ cz + d \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Some generalizations and conclusion

The idea is that if we define a relation R' as

$$R'(\vec{u}, i) \text{ iff } R(\vec{u}, i, \bar{F}_{R,(a,c),(b,d)}(\vec{u}, i))$$

then for all $0 \leq i \leq y$, we have

$$\bar{F}_{R,(a,c),(b,d)}(\vec{u}, i) = \bar{f}_{Id,R',(a,c),(b,d)}(\vec{u}, i)$$

Some generalizations and conclusion

$z = \bar{F}_{R,i,(a,c),(b,d)}(\vec{u}, lh_2(y))$ is equivalent to

$$\exists m \in \{0,1\}^{lh_2(y)}\, (\forall i)_{i \leq lh_2(y)} \left[R'(\vec{u}, i, \bar{f}_{Id,R_m,(a,c),(b,d)}(\vec{u}, i)) \leftrightarrow m_i = 1\right]$$

$$\wedge \left[z = \bar{f}_{R_m,(a,c),(b,d)}(\vec{u}, y)\right]$$

where $R_m(i)$ is define by $m_i = 1$.

Some generalizations and conclusion

Variant. The long recursion for transition function

$$h_R(a, c, b, d, \vec{u}, z, i) = \begin{cases} az + b \text{ if } R(\vec{u}, i, z) \\ cz + d \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a $\Delta_0^\sharp$-definable graph.

Some generalizations and conclusion

Consequence. The short recursion for transition function

$$h_{R,(a,c),(b,d)}(\vec{u}, i, z) = \begin{cases} a(\vec{u})z + b(\vec{u}) \text{ if } R(\vec{u}, i, z) \\ c((\vec{u})z + d((\vec{u}) \text{ if } \neg R(\vec{u}, i, z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Some generalizations and conclusion

Generalization (work in progress) The short recursion for transition function

$h_{(R_1,R_2,\ldots,R_k),(a_1,c_1),(a_2,c_2),\ldots,(a_k,c_k)}(\vec{u},i,z) =$

$$\begin{cases} a_1(\vec{u})z + b_1(\vec{u}) \text{ if } R_1(\vec{u},i,z) \\ \ldots. \\ a_k(\vec{u})z + b_k(\vec{u}) \text{ if } R_k(\vec{u},i,z) \end{cases}$$

defines a function with a Δ_0-definable graph.

Some generalizations and conclusion

Generalization of the second result. The short recursion for transition function

$$h_{a_1,b_1,a_2,b_2}(m_1, m_2, z) = (a_2\,((a_1 x + b_1) \bmod m_1) + b_2) \bmod m_2$$

defines a function with a Δ_0-definable graph.

Some generalizations and conclusion

Some generalizations and conclusion

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