Arithmetical theories and the automation of induction

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Motivation: two areas

- Mathematical logic: theories of arithmetic
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  - Empirical evaluation of implementations

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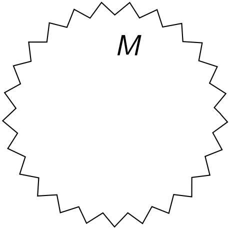

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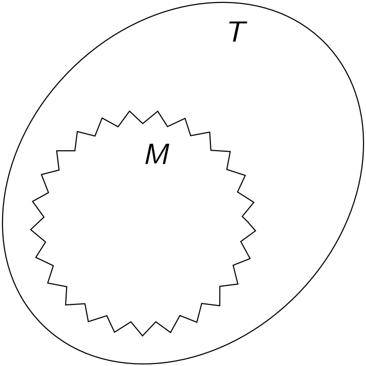

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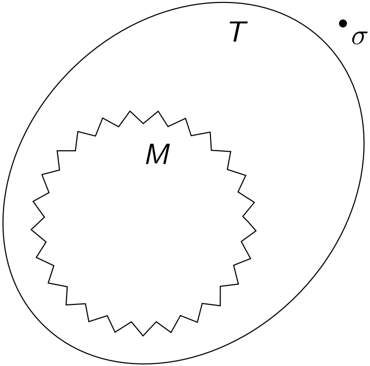

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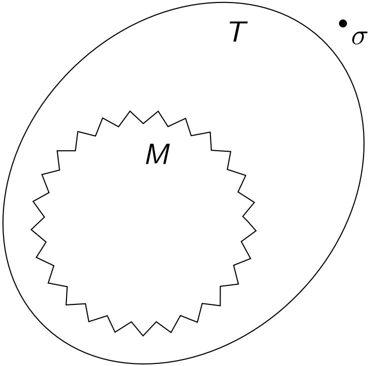

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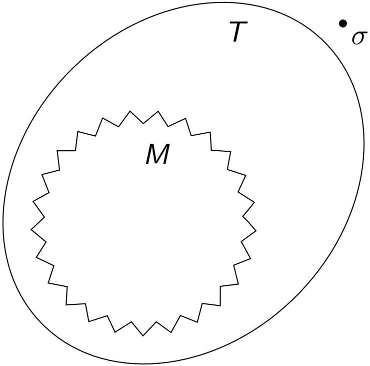

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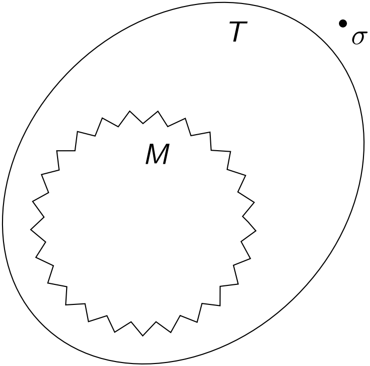

▶ Weak arithmetical theories

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- Not only numbers, also: lists, trees, etc.
- Bounded quantifiers are not distinguished
- Sometimes: idiosyncracies of method M reflected in T

Outline

Induction on Literals

- **Definition.** A *literal* is an atom or a negated atom.

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- **Definition.** The induction axiom $I_x\varphi(x, \overline{z})$ is

$$\varphi(0, \overline{z}) \land \forall x\,(\varphi(x, \overline{z}) \to \varphi(s(x), \overline{z})) \to \forall x\,\varphi(x, \overline{z}).$$

- **Definition.** For a set of formulas Γ define

$$\Gamma\text{-IND} = \{I_x\varphi(x, \overline{z}) \mid \varphi(x, \overline{z}) \in \Gamma\}$$

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- **Observation.** For natural axioms A in language $L = \{0, s, p, +\}$:

$$A + \text{Literal-IND} \equiv A + \text{Open-IND}.$$

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$$C_1, \ldots, C_n \in \mathcal{C} \text{ implies } \rho(C_1, \ldots, C_n) \in \mathcal{C}$$

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Given $\mathcal{C}$, compute closure by $\mathcal{C}^0 = \mathcal{C}, \mathcal{C}^1, \mathcal{C}^2, \ldots \longrightarrow \mathcal{C}^\omega$.
- **Definition.** $\mathcal{S}$ *sound* if $C \in \mathcal{C}^\omega$ implies $\mathcal{C} \models C$
- **Definition.** $\mathcal{S}$ *refutationally complete* if $\mathcal{C} \models \bot$ implies $\bot \in \mathcal{C}^\omega$

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- **Example.** $\mathcal{S} + \mathrm{SCIND}$ refutes

$$\{x+0=0, x+s(y)=s(x+y), c+(c+c)\neq(c+c)+c\}$$

Characterisation and independence result

- **Theorem.** $\mathcal{S}$ sound saturation system, T $\exists_2$ theory. If $\mathcal{S} + \mathsf{SCIND}$ refutes $\mathsf{CNF}(\mathsf{sk}^{\exists}(T))$ then $T + \text{Literal-IND}$ is inconsistent.

Characterisation and independence result

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- **Definition.**

  Let $T = \{\ 0 \neq s(x), s(x) = s(y) \rightarrow x = y,$
  $E(0), E(x) \rightarrow O(s(x)), O(x) \rightarrow E(s(x)), \neg E(c) \wedge \neg O(c)\}$.

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- **Theorem.** $T + \text{Literal-IND}$ is consistent.

- **Corollary.** If $\mathcal{S}$ sound saturation system, then $\mathcal{S} + \mathrm{SCIND}$ does not refute T.

Outline

Open induction

- **Theorem.** [Shoenfield '58] Over

$$
\begin{aligned}
s(x) &\neq 0 & x + 0 &= x \\
p(0) &= 0 & x + s(y) &= s(x + y) \\
p(s(x)) &= x
\end{aligned}
$$

  open induction is equivalent to

$$
\begin{aligned}
x + y &= y + x & x &= 0 \vee x = s(p(x)) \\
(x + y) + z &= x + (y + z) & x + y &= x + z \to y = z
\end{aligned}
$$

- Shepherdson '60ies: systematic study

Open induction for lists

- **Definition.** Let $L = \{\mathrm{nil}, \mathrm{cons}, \frown\}$, let $T =$

$$\begin{aligned}
\mathrm{nil} &\neq \mathrm{cons}(x, X) \\
\mathrm{cons}(x, X) = \mathrm{cons}(y, Y) &\to x = y \wedge X = Y \\
\mathrm{nil} \frown Y &= Y \\
\mathrm{cons}(x, X) \frown Y &= \mathrm{cons}(x, X \frown Y)
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- **Theorem.** [H, Vierling]

$$T + \mathit{Open}\text{-}\mathrm{IND} \nvdash Y \frown X = Z \frown X \rightarrow Y = Z.$$

Open Questions

- **Project.** Systematic picture of subsystems of open induction
  - atomic, literal, clause, dual clause, open induction
  - for numbers (in various signatures)
  - for lists, trees, . . .

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- **Definition.** An $L \cup \{\eta\}$ clause set $\mathcal{C}$ is a *clause set cycle (CSC)* if $\mathcal{C}(s(\eta)) \models \mathcal{C}(\eta)$ and $\mathcal{C}(0) \models \bot$. An $L \cup \{\eta\}$ clause set $\mathcal{D}(\eta)$ is refuted by a CSC $\mathcal{C}(\eta)$ if $\mathcal{D}(\eta) \models \mathcal{C}(\eta)$.
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- Variants
- **Example.** CSC refutes Even/Odd example

Logical characterisation

- **Definition.** Γ set of formulas, define

$$\frac{\varphi(0) \quad \varphi(x) \to \varphi(s(x))}{\varphi(\eta)} \ \Gamma\text{-}\mathsf{IND}_{\eta}^{\mathsf{R}-}$$

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- **Theorem.** $\mathcal{D}$ is refuted by a CSC iff $\mathcal{D} + [\emptyset, \exists_1\text{-IND}_{\eta}^{\mathrm{R}-}] \vdash \bot$.

Independence result

- **Definition.** $L_{\mathrm{LA}} = \{0, s, p, +\}$

  $$\begin{aligned} T = \{ \ & s(0) \neq 0, p(0) = 0, p(s(x)) = x, \\ & x + 0 = x, x + s(y) = s(x + y), \\ & x + y = y + x, x + (y + z) = (x + y) + z \ \} \end{aligned}$$

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- **Definition.** Let $k, n, m \in \mathbb{N}$ with $0 < n < m$, define $E_{k,n,m}$ as:

  $$n \cdot x + \overline{(m - n)k} = m \cdot x \rightarrow x = \overline{k}$$

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- **Example.** $x + 0 = x + x \rightarrow x = 0$ ($E_{0,1,2}$)
- **Theorem.** $T + \exists_1\text{-IND}^- \nvdash E_{k,n,m}$
- **Corollary.** $\mathcal{E}_{k,n,m}(\eta)$ is not refuted by an L_{LA} clause set cycle.

Independence result: proof

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Proof.

Countermodel $\mathcal{M}$, domain $\{(i,n) \in \mathbb{N} \times \mathbb{Z} \mid i = 0 \text{ implies } n \in \mathbb{N}\}$

$$0^{\mathcal{M}} = (0,0) \qquad p^{\mathcal{M}}((0,n)) = (0, n \dot{-} 1)$$

$$s^{\mathcal{M}}(i,n) = (i, n+1) \qquad p^{\mathcal{M}}((i,n)) = (i, n-1) \text{ if } i > 0$$

$$(i,n) +^{\mathcal{M}} (j,m) = (\max(i,j), n+m)$$

□

Open Questions

- Iterations of CSCs ... nested applications of $\exists_1\text{-IND}^{\mathsf{R}-}_{\eta}$
- **Theorem.** For every $k \in \mathbb{N}$ there is a clause set $\mathcal{C}_k$ which is refuted by $[\emptyset, \exists_1\text{-IND}^{\mathsf{R}-}]_{k+1}$ but not by $[\emptyset, \exists_1\text{-IND}^{\mathsf{R}-}]_k$
  Proof Sketch. Totality of a suitable function f_k.

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- **Conjecture.** $T_0 + \exists_1\text{-}\mathrm{IND}^{-} \nvdash x + (x + x) = (x + x) + x.$
  (Note that $T_0 + \text{Literal-IND} \vdash x + (x + x) = (x + x) + x.$)
  (Would yield corollary on CSCs)

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- Analyticity

Thank you!

Stefan Hetzl and Jannik Vierling.
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