Non-elementary cofinal extensions of models of fragments of Peano arithmetic

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Ongoing joint work with Wong Tin Lok

This talk:

- Focus on cofinal extensions for models of strong fragments of arithmetic, especially the elementarity of extensions.
- Motivating Question: What 'controls' the elementarity of a cofinal extension?
  Answer: Second-order theory of a definable cut。

Plan:

- Introduction.
- Non-elementary cofinal extension for countable model.
- A systematic way to 'compress' truth.

First-order Arithmetic

- The language of First-order Arithmetic
  $\mathcal{L}_1 = \{+, \times, <, =, 0, 1\}$.
- A formula is Δ_0 if all its quantifiers are bounded
  i.e., in the form of $\exists x < t(\overline{y})\ \phi(x, \overline{y})$ or $\forall x < t(\overline{y})\ \phi(x, \overline{y})$.
- $\Sigma_n = \{\exists \overline{x_1}\, \forall \overline{x_2}\ \ldots Q\overline{x_n}\, \theta(\overline{x_1}, \ldots \overline{x_n}, \overline{a}) \mid \theta \in \Delta_0\}$,
  $\Pi_n = \{\forall \overline{x_1}\, \exists \overline{x_2}\ \ldots Q\overline{x_n}\, \theta(\overline{x_1}, \ldots \overline{x_n}, \overline{a}) \mid \theta \in \Delta_0\}$.
- A formula is Δ_n if it is equivalent to both a Σ_n and a Π_n formula(over some model or theory).
- $\mathrm{I}\Sigma_n$ consists of PA$^-$ and the **Induction** for all Σ_n formula ϕ:

$$\phi(0, \overline{c}) \wedge (\forall x(\phi(x, \overline{c}) \to \phi(x+1, \overline{c})) \to \forall x \phi(x, \overline{c}).$$

- $\mathrm{B}\Sigma_n$ consists of $\mathrm{I}\Sigma_0$ and the **Collection** for all Σ_n formula ϕ:

$$\forall x < a\ \exists y \phi(x, y, \overline{c}) \to \exists b \forall x < a\ \exists y < b\ \phi(x, y, \overline{c}).$$

- $\exp$ asserts the totality of exponential function.
- PA$=\bigcup_{n \in \omega} \mathrm{I}\Sigma_n$.

First-order Arithmetic

Theorem (Paris–Kirby 1978)

$$\mathrm{I}\Delta_0 + \mathsf{exp} \dashv \mathrm{B}\Sigma_1 + \mathsf{exp} \dashv \mathrm{I}\Sigma_1 \dashv \mathrm{B}\Sigma_2 \dashv \mathrm{I}\Sigma_2 \dashv \mathrm{B}\Sigma_3 \ldots$$

and none of the converses holds.

Definition

We call models of $\mathrm{B}\Sigma_n + \neg\mathrm{I}\Sigma_n$ ***B-Models***, and models of $\mathrm{I}\Sigma_n + \neg\mathrm{B}\Sigma_{n+1}$ ***I-Models***.

Cofinal Extensions vs Elementarity

Definition

Let $M, K \models \mathrm{PA}^-$, $M \subseteq K$

- K is a **cofinal extension** of M ($M \subseteq_{\mathrm{cf}} K$) if

$$\forall x \in K \quad \exists y \in M \quad K \models x < y.$$

  The extension is proper if $K \neq M$.

- For each $n \in \omega$, K is a **n-elementary extension** of M ($M \preccurlyeq_n K$) if for every $\phi \in \Sigma_n$, $\overline{a} \in M$,

$$M \models \phi(\overline{a}) \iff K \models \phi(\overline{a}).$$

Convention: We assume all models mentioned satisfy $\mathrm{I}\Delta_0 + \exp$. Since $\mathrm{I}\Delta_0 + \exp \vdash \mathrm{MRDP}$, we have 0-elementarity for any extension.

Cofinal Extensions vs Elementarity

Theorem (Gaifman–Dimitracopoulos 1980)

Let $n \in \omega$, $M, K \models \mathrm{PA}^-$, $M \subseteq_{\mathrm{cf}} K$, *if* $M \models \mathrm{B}\Sigma_{n+1}$, *then* $M \preccurlyeq_{n+2} K$. *In particular, if* $M \models \mathrm{PA}$, *then* $M \preccurlyeq K$.

Theorem (Kaye 1991)

- *For each* $n \in \omega$, *every countable model of* $\mathrm{B}\Sigma_{n+1} + \exp + \neg \mathrm{I}\Sigma_{n+1}$ *admits a proper elementary cofinal extension.*
- *(Informally) Every sufficiently saturated countable model admits a proper elementary cofinal extension.*

Question: How much non-elementarity can we get for cofinal extensions?

Non-elementary Cofinal Extension

Theorem

For every countable model $M \models \mathrm{I}\Delta_0 + \exp$, if $M \not\models \mathrm{PA}$ then M admits a non-elementary cofinal extension.

- (Paris 1981, Friedman independently.) Every countable model $M \models \mathrm{I}\Sigma_n + \exp + \neg\mathrm{B}\Sigma_{n+1}$ admits a cofinal extension $K \models \mathrm{B}\Sigma_{n+1}$. So we only need to consider B-models. $(M \models \mathrm{B}\Sigma_{n+1} + \neg\mathrm{I}\Sigma_{n+1}.)$
- We can further require that there is $M \preccurlyeq L$ such that $M \subseteq_{\mathrm{cf}} K \subseteq_{\mathrm{e}} L$ and $M \not\preccurlyeq K$. This answers a question negatively in Kaye's paper 'On Cofinal Extensions of Models of Fragments of Arithmetic'(1991).

Corollary

A countable model of $\mathrm{I}\Delta_0 + \exp$ satisfy PA if and only if all of its cofinal extensions are fully elementary.

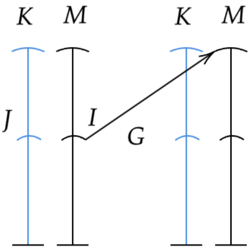
What Does $\neg\mathrm{I}\Sigma_{n+1}$ Provide Us?

Fact

If $M \models \neg\mathrm{I}\Sigma_{n+1}$, then:

- There is a Σ_{n+1}-definable proper cut $I \subseteq_e M$.
- There is a $\Sigma_n \wedge \Pi_n$-definable non-decreasing total function $G\colon I \to M$, whose range is cofinal in M.

Suppose $M \models \mathrm{B}\Sigma_{n+1}$. Let $\phi(x, y) \in \Sigma_n \wedge \Pi_n$ defines the graph of G, then for any $K \supseteq_{\mathrm{cf}} M$, $\exists y\, \phi(x, y)$ defines a cut $J \supseteq_{\mathrm{cf}} I$.
In particular, if ω is Σ_{n+1}-definable in M then it is still Σ_{n+1}-definable in any cofinal extension.



Non-elementary Cofinal Extension for B-Model

Fix $n \in \omega$, $M \models \mathrm{B}\Sigma_{n+1} + \exp + \neg\mathrm{I}\Sigma_{n+1}$.
$\mathrm{SSy}_I(M) = \{A \subseteq I \mid A \text{ coded in } M\}$, $\mathrm{SSy}(M) = \mathrm{SSy}_\omega(M)$.

Observation

If ω is Σ_{n+1}-definable in M, then we may describe the second-order property of $(\omega, \mathrm{SSy}(M))$ using first-order formula in M.
e.g: $0' \in \mathrm{SSy}(M)$ can be described as

$$\exists a\, \forall i \in \omega\, (i \in a \leftrightarrow \exists s \in \omega\, (M_i(i) \textit{ halts with computation } s)).$$

Here $M_i(i)$ means the i-th Turing Machine with input i.

Cofinally extend M to K such that $\mathrm{SSy}(M)$ and $\mathrm{SSy}(K)$ satisfy different second-order properties.

Non-elementary Cofinal Extension for $\mathbb{B}$-Model

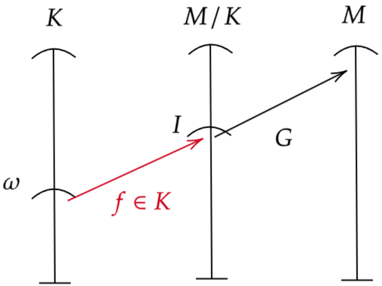
Lemma

Every countable $M \models \neg \mathrm{I}\Sigma_{n+1}$ admits a cofinal extension in which ω is Σ_{n+1}-definable.

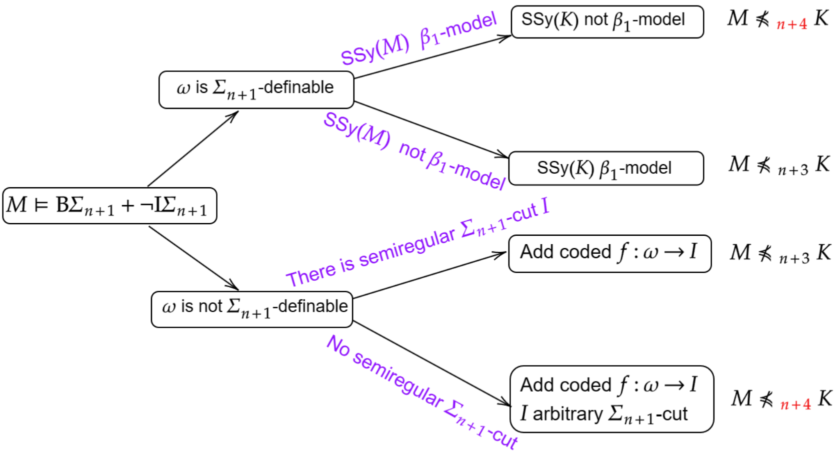
Proof Sketch: If ω is not Σ_{n+1}-definable in M, add a coded non-decreasing function into $K \supseteq_{\mathrm{cf}} M$:

$$f : \omega \to I$$

where I is a Σ_{n+1}-definable cut in M, and $\mathrm{Range}(f) \subseteq_{\mathrm{cf}} I$.
Now ω is Σ_{n+1}-definable in K, and I becomes non-semiregular.



Non-elementary Cofinal Extension for B-Model



Here $\mathrm{SSy}(M)$ is a β_1-model means $(\omega, \mathrm{SSy}(M)) \preccurlyeq_{\Sigma^1_1} (\omega, \mathcal{P}(\omega))$.
Question: Are the non-elementarities here optimal?

Non-elementary Cofinal Extension for B-Model

Theorem

For each $n \in \omega$*:*

- *There is a countable* $M \models \mathrm{B}\Sigma_{n+1} + \exp + \neg \mathrm{I}\Sigma_{n+1}$*, such that for any cofinal extension* $K \supseteq_{\mathrm{cf}} M$*,* $M \preccurlyeq_{n+3} K$*.*
- *There is a uncountable* $M \models \mathrm{B}\Sigma_{n+1} + \exp + \neg \mathrm{I}\Sigma_{n+1}$*, such that for any cofinal extension* $K \supseteq_{\mathrm{cf}} M$*,* $M \preccurlyeq K$*.*

Compressing the Truth

We just mentioned that we can describe the second-order property of $(I, \mathrm{SSy}_I(M))$ in M if I is definable.
Actually, the converse is also true: For fixed parameter, we can describe the first-order truth of M in $(I, \mathrm{SSy}_I(M))$ if I is Σ_{n+1}-definable.

Two ingredients for compressing

- $\neg \mathrm{I}\Sigma_{n+1}$: Cofinal function $G\colon I \to M$.
- $\mathrm{B}\Sigma_{n+1}$: Encoding Δ_{n+1} formulas.

Encoding Δ_{n+1} over a Cut

Definition (Δ_{n+1} over a Cut)

We say that $\alpha(\overline{x}, \overline{i})$ is Δ_{n+1} over $\overline{i} \in I$, if $\alpha \in \Sigma_{n+1}$ and there is some $\beta \in \Pi_{n+1}$ such that

$$M \models \forall x \forall \overline{i} \in I(\alpha(\overline{x}, \overline{i}) \leftrightarrow \beta(\overline{x}, \overline{i})).$$

Theorem (Chong–Mourad Coding Lemma 1990)

If $M \models \mathrm{B}\Sigma_{n+1} + \exp$ and I is a Σ_{n+1}-definable cut of M, $\alpha(\overline{x}, \overline{i})$ is Δ_{n+1} over $\overline{i} \in I$, then for all $b \in M$

$$M \models \exists c\,(\forall x < b\,\forall \overline{i} \in I(\langle \overline{x}, \overline{i} \rangle \in c \leftrightarrow \alpha(\overline{x}, \overline{i})).$$

Encoding Δ_{n+1} over a Cut

We also have $\mathrm{B}\Sigma_{n+1}$ over a Σ_{n+1} definable cut.

Theorem (Belanger–Chong–Li–Wong–Yang)

If $M \models \mathrm{B}\Sigma_{n+1} + \exp$ and I is a Σ_{n+1}-definable cut of M, $\alpha(x, i)$ is Δ_{n+1} over $i \in I$, then for all $a \in M$

$$M \models \forall x < a \, \exists i \in I \, \alpha(x, i) \to \exists b \in I \, \forall x < a \, \exists i < b \, \alpha(x, i).$$

Rewriting System

For simplicity, we assume that $n = 0$ and $I = \omega$. We fix some $M \models \mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$, so $G \colon \omega \to M$ is Δ_0-definable.

Observation 1

For any Π_2 formula $\forall a\, \exists b\, \theta(a, b)$ where $\theta \in \Delta_0$:

$$
\begin{aligned}
& \forall a\, \exists b\, \theta(a, b) \\
\Leftrightarrow\quad & \forall x \in \omega \exists y \in \omega\, \underbrace{\forall a < G(x) \exists b < G(y) \theta(a, b)}_{\alpha(x,y) \in \Delta_1 \text{ over } x,y \in \omega} && \text{G is Cofinal} \\
\Leftrightarrow\quad & \exists f \colon \omega \to \omega\, \forall x \in \omega\, \alpha(x, f(x)) && \text{Coding Lemma}
\end{aligned}
$$

$f \colon \omega \to \omega$ means f codes the graph of a total function from ω to ω.

Rewriting System

Temporarily let $\exists f$ abbreviates $\exists f : \omega \to \omega$.

Observation 2

For any Σ_3 formula, it is equivalent to:

$$\begin{aligned}
& \exists c \forall a \exists b\, \theta(a, b, c) \\
\Leftrightarrow\ & \exists c \exists f \forall x \in \omega\, \alpha(x, f(x), c) && \text{Observation 1} \\
\Leftrightarrow\ & \exists z \in \omega \exists f\, \exists c < G(z) \forall x \in \omega\, \alpha(x, f(x), c) && \text{G is Cofinal} \\
\Leftrightarrow\ & \exists z \in \omega \exists f\, \forall x' \in \omega\, \underbrace{\exists c < G(z) \forall x < x'\, \alpha(x, f(x), c)}_{\beta(z, x', f\restriction_{x'}) \in \Delta_1 \text{ over } z, x', f\restriction_{x'} \in \omega} && \mathrm{B}\Sigma_1 \text{ over } \omega
\end{aligned}$$

Here fixing $z \in \omega$, those $x, f\restriction_x \in \omega$ satisfy $\beta(z, x, f\restriction_x)$ provides us an ω-branching tree.

Normal Form

With similar rewriting procedure(although more tedious!), we can show the following:

Theorem (Normal Form Theorem)

Let $m \in \omega$, $M \models \mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$ and ω is Σ_1-definable in M. Then any Σ_{m+3} formula $\phi(\overline{c})$ in M is equivalent to the form:

$$\exists f_1 \, \forall f_2 \ldots Q f_m \, \overline{Q} x \in \omega \, \alpha(\overline{f}\restriction_x, x, \overline{c})$$

for some $\alpha \in \Delta_1$ over $\overline{f}\restriction_x, x \in \omega$ effectively decided by ϕ, and such equivalence is provable in $\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$.

A Bit Second-order Arithmetic

Definition

A second-order formula is $\mathrm{r}\Sigma^1_m$ if it has the form

$$\exists f_1 \, \forall f_2 \, \dots Q f_m \, \overline{Q} x \, S(x, \overline{f}\restriction_x)$$

where $S \in \Delta^0_0$ and $\exists f$ abbreviates $\exists f : \omega \to \omega$.

- $\mathrm{r}\Sigma^1_m$ and Σ^1_m coincide over ACA_0.
- For extensions of second-order structure, we write $(\omega, \mathcal{A}) \preccurlyeq_{\mathrm{r}\Sigma^1_m} (\omega, \mathcal{B})$ if $\mathrm{r}\Sigma^1_m$ formulas are absolute.

Correspondence Theorem

Lemma (Belanger–Wong)

Let $M \models \mathrm{B}\Sigma_1 + \exp$, $M \subseteq_{\mathrm{cf}} K$, *then* $K \models \mathrm{B}\Sigma_1 + \exp$.

So the same normal form applied in M and K.

Theorem (First-order Second-order Correspondence)

For $M \models \mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$, $M \subseteq_{\mathrm{cf}} K$, ω *is* Σ_1*-definable in* M, *let* $m \in \omega$,

$$M \preccurlyeq_{m+3} K \iff (\omega, \mathrm{SSy}(M)) \preccurlyeq_{\mathrm{r}\Sigma^1_{m+1}} (\omega, \mathrm{SSy}(K)).$$

The correspondence helps us to convert a problem of first-order arithmetic into a problem of second-order arithmetic.

β_m-model

Definition (β_m-model)

For $m > 0$, a second-order structure $(\omega, \mathcal{A})$ is called a β_m-model if

$$(\omega, \mathcal{A}) \preccurlyeq_{\Sigma^1_m} (\omega, \mathcal{P}(\omega)).$$

Theorem (Mummert–Simpson 2004)

For each $m > 0$, there is a countable β_m-model which is not a β_{m+1}-model.

Constructions of Cofinal Extension

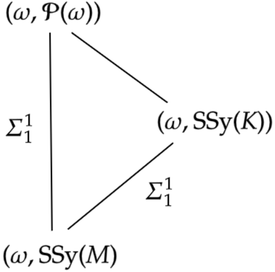
For models of $\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$ in which ω is Σ_1 definable:

- If $M \subseteq_{\mathrm{cf}} K$ and $\mathrm{SSy}(M) = \mathrm{SSy}(K)$, then $M \preccurlyeq K$.
- For each $m \in \omega$, there is a countable $M \subseteq_{\mathrm{cf}} K$ such that $M \preccurlyeq_{m+2} K$ but $M \not\preccurlyeq_{m+3} K$.
  $((\omega, \mathrm{SSy}(M))$ is a β_m-model but not a β_{m+1}-model.)

Constructions of Cofinal Extension

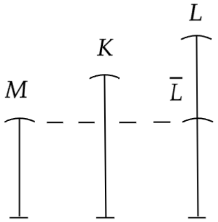
For models of $\mathrm{B}\Sigma_1 + \exp + \neg\mathrm{I}\Sigma_1$:

- Every model M_0 admits an uncountable cofinal extension $M_0 \subseteq_{\mathrm{cf}} M$, such that every cofinal extension of M is fully elementary. ($\mathrm{SSy}(M) = \mathcal{P}(\omega)$.)
- Every countable M_0 admits a countable cofinal extension $M_0 \subseteq_{\mathrm{cf}} M$, such that any cofinal extension of M is 3-elementary. ($(\omega, \mathrm{SSy}(M))$ is a β_1-model.)



Constructions of Cofinal Extension

There is a countable M, such that for any extension $M \subseteq K$, there is a further elementary extension $K \preccurlyeq L$ such that $M \preccurlyeq_{\mathrm{cf}} \overline{L} \subseteq_{\mathrm{e}} L$. $\quad ((\omega, \mathrm{SSy}(M)) \preccurlyeq (\omega, \mathcal{P}(\omega).))$



This answers a main question positively in Kaye's another paper 'Model-theoretic properties characterizing Peano arithmetic'(1991).

One of the most annoying failures of this paper is the absence of a result similar to 3.7 showing (ii) to be insufficient to characterize PA over $\mathrm{I}\Delta_0 + \exp$ on its own.

PROBLEM 4.2. For all $n \geq 1$, is there a countable model $I_n \models \mathrm{I}\Delta_0 + \exp + \mathrm{B}\Sigma_n + \neg \mathrm{I}\Sigma_n$ such that whenever $L \succ I_n$ is countable there is $\bar{L} \succ L$ and $\bar{K} \subseteq_{\mathrm{e}} \bar{L}$ such that $I_n \prec_{\mathrm{cf}} \bar{K}$?

Answering Kaye's Questions

In the same paper, Kaye considers various model-theoretic conditions, and asks whether they characterize arithmetic theories extending PA. Here we can show that most of them fail to rule out the case of extending $\mathrm{B}\Sigma_{n+1} + \neg\mathrm{I}\Sigma_{n+1}$.

There are various natural variations on property (ii) of a theory T extending $\mathrm{I}\Delta_0 + \exp$ that have been considered in preliminary drafts of this paper. These include:

(iii) For each complete consistent $S \supseteq T$ there is a model $K \models S$ such that whenever $L \succ K$ there is $M \subseteq_e L$ with $K \prec_{cf} M$.

(iv) For each complete consistent $S \supseteq T$ there is $K \models S$ such that whenever $L \succ K$ (no restriction on card(L)) there is $\bar{L} \succ L$ and $\bar{K} \subseteq_e \bar{L}$ with $K \prec_{cf} \bar{K}$.

(v) For each consistent $S \supseteq T$ there are models K, L, M of S such that $K \prec_{cf} M \subseteq_e L$ with $M \neq L$ and $K \prec L$.

(vi) For each consistent $S \supseteq T$ there is a consistent $S^* \supseteq S$ such that whenever $K \subseteq_{cf} L$ are both models of S^*, then $K \prec L$.

Each of these properties is true of the theory $T = \mathrm{PA}$; moreover, if T has any one of (iii), (iv) or (v), then $T \vdash \mathrm{I}\Sigma_n \Rightarrow T \vdash \mathrm{B}\Sigma_{n+1}$ for each $n \in \mathbb{N}$.

PROBLEM 4.3. If T extends $\mathrm{I}\Sigma_n + \exp$ and satisfies (vi), does $T \vdash \mathrm{B}\Sigma_{n+1}$? Indeed, what theories $T \supseteq \mathrm{I}\Delta_0 + \exp$ (other than extensions of PA) have property (vi)?

PROBLEM 4.4. Are there theories T extending $\mathrm{B}\Sigma_{n+1} + \exp + \neg\mathrm{I}\Sigma_{n+1}$ for some $n \in \mathbb{N}$ satisfying (iii), (iv), (v) or (vi)?

Generalized Correspondence

Theorem (First-order Second-order Correspondence, Generalized)

For $n \in \omega$, suppose $M \models \mathrm{B}\Sigma_{n+1} + \exp + \neg\mathrm{I}\Sigma_{n+1}$ with a Σ_{n+1}-definable cut I which is closed under exponentiation in M. $M \subseteq K$ is a $(n+2)$-elementary extension and $K \models \mathrm{B}\Sigma_{n+1}$. Let J be the Σ_{n+1}-definable cut in K with the same definition as M, then $(I, \mathsf{SSy}_I(M))$ naturally embed into $(J, \mathsf{SSy}_J(K))$, and for all $m \in \omega$:

$$M \preccurlyeq_{n+m+3} K \iff (I, \mathrm{SSy}_I(M)) \preccurlyeq_{\mathrm{r}\Sigma^1_{m+n+1}} (J, \mathrm{SSy}_J(K)).$$

Summary

- Any countable model of $\mathrm{I}\Delta_0 + \exp$ fail to satisfy PA admits a non-elementary cofinal extension.
- A systematic way to 'compress' truth in $M \models \mathrm{B}\Sigma_{n+1} + \neg\mathrm{I}\Sigma_{n+1}$ in the second-order theory of its Σ_{n+1}-definable cut.
- For the case ω is Σ_{n+1}-definable, we construct models with various cofinal extension properties by considering its standard system.

Summary

- Any countable model of $\mathrm{I}\Delta_0 + \exp$ fail to satisfy PA admits a non-elementary cofinal extension.
- A systematic way to 'compress' truth in $M \models \mathrm{B}\Sigma_{n+1} + \neg\mathrm{I}\Sigma_{n+1}$ in the second-order theory of its Σ_{n+1}-definable cut.
- For the case ω is Σ_{n+1}-definable, we construct models with various cofinal extension properties by considering its standard system.

Thank You!