On Automorphisms of Short Models of PA

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A number of results regarding the automorphism group of a countable arithmetically saturated model of PA has been proven.

Kossak-Schmerl 95 proves that the automorphism group of a countable arithmetically saturated model of PA codes its standard system. Using that result they prove:

Theorem

Let M_1, M_2 be countable arithmetically saturated models of Peano Arithmetic such that $\mathrm{Aut}(M_1) \cong \mathrm{Aut}(M_2)$. Then $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$.

N 06 shows that if M is a countable arithmetically saturated of Peano Arithmetic, then $\mathrm{Aut}(M)$ can recognize if some maximal open subgroup is a stabilizer of a nonstandard element, which is smaller than any nonstandard definable element. That fact is used to show

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Here RT^n_2 is Infinite Ramsey's Theorem stating that every 2-coloring of $[\omega]^n$ has an infinite homogeneous set.

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Here RT^n_2 is Infinite Ramsey's Theorem stating that every 2-coloring of $[\omega]^n$ has an infinite homogeneous set.
We will outline similar results for the automorphism group of a countable short arithmetically saturated models of PA.

Let M be a model of Peano Arithmetic and let $G = \mathrm{Aut}(M)$.
If $A \subset M$ we use the notation $G_{(A)} = \{g \in G | \forall x \in A : g(x) = x\}$ and $G_{\{A\}} = \{g \in G | g(A) = A\}$.

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Pointwise stabilizers of finite sets (subgroups of the form $G_{(A)}$ for some finite $A \subset M$) are the basic open subgroups of $\mathrm{Aut}(M)$.

We define
$\mathrm{SSy}(M) = \{X \subseteq \omega | X = Y \cap \omega$ for some Y definable in $M\}$ and
$\mathrm{Rep}(T) = \{\omega \cap X : X$ is a definable set in the prime model of $T\}$

The model M is called *recursively saturated* if it realizes all recursive types $p(x, a), a \in M$.
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The standard cut ω is *strong* in $M \models \mathrm{PA}$ if for every $a \in M$ there is $c > \omega$ such that for every $i \in \omega, (a)_i \in \omega \leftrightarrow (a)_i < c$.

Theorem (Kirby 84)

If M is a countable recursively saturated model of Peano Arithmetic, then the following are equivalent:

1. *M is arithmetically saturated;*
2. *ω is strong in M;*
3. $(\omega, \mathrm{SSy}(M)) \models \mathrm{ACA}_0$.

Definition

A type $p(v, \bar{a})$ over a model M is *bounded* if it contains the formula $v < t(\bar{a})$ for some Skolem term t. A model M is *short recursively saturated* if and only if M is not recursively saturated and every bounded recursive type $p(v, \bar{a})$, with $\bar{a} \in M$, is realized in M.

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Let M be a model of PA and $a \in M$. Let $M(a) = \{b \in M : b < t(a) \text{ for some Skolem term } t\}$.

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Theorem (Smoryński 81)

A model N is short recursively saturated if and only if $N \cong M(a)$ for some recursively saturated model M and $a \in M$.

Let I be a cut in a model M. We say I is *upward invariant* if there is a sequence of definable elements in M which is upward cofinal in I. We say I is *downward invariant* if there is a sequence of definable elements in M which is downward cofinal in I. We say I is *invariant* if I is either upward or downward invariant.

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It is not difficult to see that if I is an invariant cut, then $\mathrm{Aut}(M)_{(I)}$ is a closed normal subgroup in $\mathrm{Aut}(M)$. Kaye 94 shows that in countable recursively saturated models of PA the converse is true. Shochat 10 shows that in countable short recursively saturated models of PA the converse is true.

Let M be a model of PA. We define Ω_ω to be the set of all elements greater than the standard cut and smaller than any nonstandard definable element. If $\Omega_\omega \neq \emptyset$ then Ω_ω is called the *smallest interstice.*

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Lemma

Let $M(a)$ be a countable short recursively saturated model of Peano Arithmetic such that Ω_ω is nonempty. Then $\mathrm{Aut}(M(a))_{(\Omega_\omega)}$ is the largest closed normal subgroup of $\mathrm{Aut}(M(a))$.

If Ω_ω is nonempty, we let

$$\mathfrak{R} = \{g | g : M \to M \text{ is a definable function such that for every } a \le b \in \Omega_\omega,\ g(a) \le g(b) \in \Omega_\omega\}.$$

If $a \in \Omega_\omega$, we define

$$\mathrm{igap}(a) = \{b \in \Omega_\omega : a \le f(b),\ \text{and } b \le f(a) \text{ for some } f \in \mathfrak{R}\}.$$

$\mathrm{igap}(a)$ is called an *intersticial gap* of the interstice Ω_ω.
When working with Ω_ω we say that the cut $I \subseteq \omega \cup \Omega_\omega$ is an *icut* if I is closed under $\mathfrak{R}$.

For the rest of this talk we fix $M(a)$ to be a countable short recursively saturated model of Peano Arithmetic (PA) and $G(a)$ to be its automorphism group, $G(a) = \mathrm{Aut}(M(a))$.

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Definition

Given subgroups H, K of $G(a)$, we say that H *precedes* K if $\bigcap_{h \in H} K^h \leq G(a)_{(\Omega_\omega)}$ (where $K^h = h^{-1}Kh$).

Definition

Given a subgroup H of $G(a)$, we define two subgroups of $G(a)$:

1. H_* is the intersection of all conjugates of H which precede H;
2. H^* is the closure of the subgroup generated by the union of all K_*, where K is a conjugate of H which is preceded by H.

Definition

A subgroup H of $G(a)$ is called *nice* if it satisfies the following properties:

1. H is a maximal open subgroup of $G(a)$;
2. $G(a)_{(\Omega_\omega)} < H$;
3. for every $f \in G(a)$, if $H^f \neq H$ then either H^f precedes H or H precedes H^f;
4. whenever K is an open subgroup of H, then $K \cap H_* > K \cap G(a)_{(\Omega_\omega)}$;
5. H_* is a closed normal subgroup of H.

Lemma

If H is a nice subgroup of $G(a)$, then $H = G(a)_{\{J\}}$ for some icut $J \subseteq_e \omega \cup \Omega_\omega$.

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Lemma

Let H be a nice subgroup of $G(a)$. If $H = G(a)_{\{J\}}$ where J is an icut in $\omega \cup \Omega_\omega$ such that $J \neq \sup(D)$ for any igap D, then $H_* = G(a)_{(J)}$.

Theorem

Let H be a nice subgroup of $G(a)$ such that there is a normal closed subgroup N of H with $H^* < N < H_*$. Then H is a stabilizer of a point from Ω_ω.

Theorem shows that if $M(a)$ is a countable short recursively saturated of Peano Arithmetic, then the topological group $\mathrm{Aut}(M(a))$ can recognize if a nice subgroup is a stabilizer of a nonstandard element, which is smaller than any nonstandard definable element.

Definition

Let H be a subgroup of $G(a)$, we say that H is n-indiscernible if:

1. H is nice;
2. H is a stabilizer of a point from Ω_ω;
3. if $H_1, \ldots, H_n$ and $K_1, \ldots, K_n$ are subgroups isomorphic to H such that H_i precedes H_j and K_i precedes K_j whenever $1 \leq i < j \leq n$ then there exists $g \in G$ such that $gK_ig^{-1} = H_i$ for all $1 \leq i \leq n$.

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If $b \in \Omega_\omega(M(a))$ then we say $\mathrm{tp}(b)$ is n-indiscernible iff b is nondefinable and for every $a_0 < a_1 < \ldots < a_{n-1} \in M(a)$ and $b_0 < b_1 < \ldots b_{n-1} \in M$, if $\mathrm{tp}(a_0) = \ldots = \mathrm{tp}(a_{n-1}) = \mathrm{tp}(b)$ and $\mathrm{tp}(b_0) = \ldots = \mathrm{tp}(b_{n-1}) = \mathrm{tp}(b)$, then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$.

Lemma

Let H be a subgroup of $G(a)$. Then H is n-indiscernible if and only if $H = G_b$ for some unbounded $b \in \Omega_\omega(M(a))$ such that $\mathrm{tp}(b)$ is n-indiscernible, $n \geq 2$.

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Corollary

Let H be a subgroup of $\mathrm{Aut}(M(a))$. Then the topological group $\mathrm{Aut}(M(a))$ recognizes if $H = G_b$ for $b \in M$ such that $\mathrm{tp}(b)$ is unbounded and n-indiscernible, $n \geq 2$.

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Theorem

Let M_1, M_2 be countable short arithmetically saturated models of PA. If $\mathrm{Aut}(M_1), \mathrm{Aut}(M_2)$ are topologically isomorphic. Then for every $n < \omega$

$$(\omega, \mathrm{Rep}(\mathrm{Th}(M_1))) \models \mathrm{RT}^n_2 \text{ iff } (\omega, \mathrm{Rep}(\mathrm{Th}(M_2))) \models \mathrm{RT}^n_2.$$

We say $b \in \Omega_\omega(M(a))$ realizes a *rare type* if b is the only element realizing $\mathrm{tp}(b)$ in $\mathrm{igap}(b)$. An igap D is called *labeled*, if there is $b \in D$ such that $\mathrm{tp}(b)$ is a rare type.

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Definition

Given subgroups $K, H_0, H_1, H_2, \ldots$ of $G(a)$, we say that K supports $\langle H_0, H_1, H_2, \ldots \rangle$ if the following hold:

1. K is a stabilizer of a labeled igap;
2. $H_i, i < \omega$ is 2-indiscernible;
3. $H_i, H_j, i < j < \omega$ are isomorphic and H_i precedes H_j;
4. if $K \leq H < G(a)$ then $H \in \{K, H_0, H_1, H_2, \ldots\}$;
5. $K < H_i, i < \omega$.

Definition

Let $X \subseteq \omega$. Then $G(a)$ encodes X if either X is finite or $X = \{i_0, i_1, i_2, \ldots\}$ where $i_0 < i_1 < i_2 < \ldots$, and there are subgroups $K_1, K_2, H_0, H_1, H_2, \ldots$ of $G(a)$ such that K_1 supports $\langle H_0, H_1, H_2, \ldots \rangle$ and K_2 supports $\langle H_{i_0}, H_{i_1}, H_{i_2}, \ldots \rangle$.

Theorem

Let M_1, M_2 be countable arithmetically saturated models of Peano Arithmetic such that $(\omega, \mathrm{Rep}(\mathrm{Th}(M_1))) \models \mathrm{RT}^2_2$. If $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$ are topologically isomorphic then $\mathrm{SSy}(M_1) = \mathrm{SSy}(M_2)$.