A study of the stability regions of systems of coupled symplectic maps

Thomas Tziotzios, Vassilis Koukouloyannis and George Voyatzis
School of Physics, AUTh, e-mail:vkouk@physics.auth.gr, voyatzis@auth.gr

Abstract. We study the stability of localized oscillations in an spatially extended system (Discrete Breathers or DBs) of weakly coupled symplectic maps. These oscillations correspond to isolated periodic orbits of the system and a linear stability analysis is performed. The phase space of the system and the non linear stability of the oscillations is studied by using a scanning method, based on the calculation of FLI, of specific phase planes and the distinction between areas with regular and chaotic motion. Finally, we examine the possibility of using the scanning method in order to locate initial conditions for such motions.

I. Intoduction

We consider the **integrable** symplectic **Suris** map which is described by the equations

$$x' = x + 4\pi^2 y', \quad y' = y + V'(x)$$

with

$$V'(x) = -\frac{1}{4\pi^2}\arctan\left(\frac{\delta\sin(x)}{1+\delta\cos(x)}\right)$$

The phase diagram of the map is the one of fig. 1 and its integral is

$$\Phi(x,y) = \cos(2\pi^2 y) + \delta\cos(x - 2\pi^2 y)$$

and $\delta = 1/3$

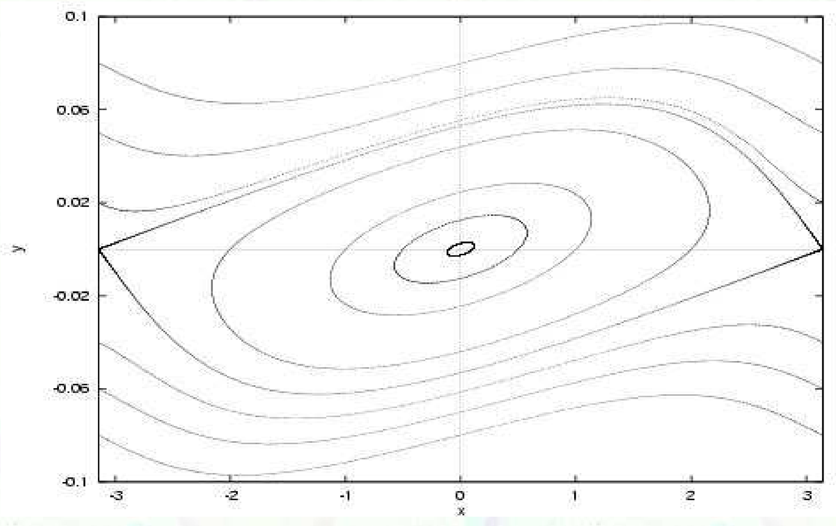

Fig.1: Phase diagram of the symplectic Suris map

We construct a chain of coupled Suris maps with a weak nearest neighbor coupling and end up with the system

$$x'_k = x_k + 4\pi^2 y'_k$$
$$y'_k = y_k + V'(x_k) + \varepsilon\sin(x_{k+1} - x_k) + \varepsilon\sin(x_{k-1} - x_k) \qquad k \in \mathbb{Z}$$

Where k is the index of the oscillator. We consider chains of differrent size and their edges always at rest $(x,y)=(0,0)$

II. The scan method of the phase space and the stability specification

In order to study the behavior of the system in the region of a periodic orbit for different values of the perturbation and for different lenghts of the chain we shall use the chaoticity index FLI (Fast Lyapunov Indicator) which is defined by

$$FLI = \sup\left(\log\left(\|z_n\|\right)\right)$$

where z_n is the deviation vector at the n-th step of the map. We consider planes in the phase space which are defined by the initial conditions (x,y) of the central oscillator and fixed initial conditions for the rest of them. We evolve the orbits of the system which correspond to each initial condition (for a maximum number of iterations of 15000) and compute the FLI. An orbit is consider as chaotic (and it is denoted with blue color in the scan plane) when the FLI comes over a critical value (20 in the particular case). The grades in the red color denote the different values of FLI when the predetermined number of iterations (15000) is realized and denote regular-stable orbits.

III. Numeric Results

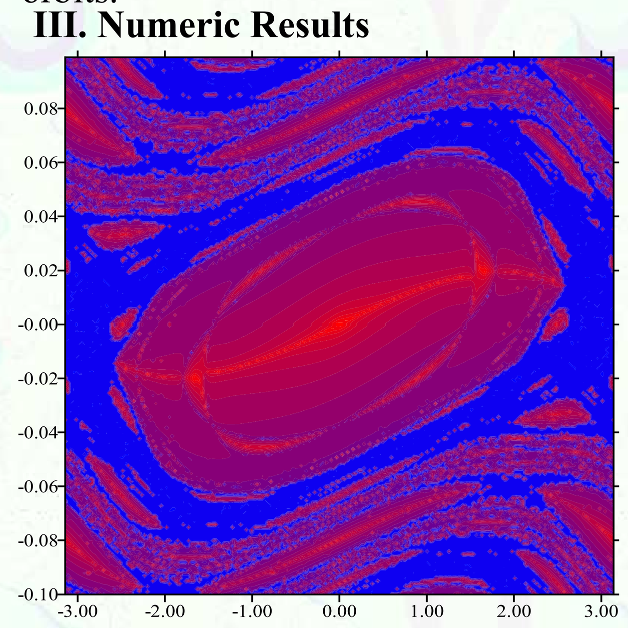

Fig. 2: Chaotic and regular orbits in a chain of length k=3 for ε =0.0028

At first we consider a chain of 3 oscillators. In this case only one of them is free since, as we have already mentioned, the oscillators of the edges are fixed at (0,0). This oscillator gets initial conditions which scan all the allowed region. This way we get fig. 2.

Note the direct relation between fig. 2 and the phase diagram of the single oscillator. We note also the creation of the chains which correspond to the isolated periodic orbits of the system. We choose the chain which corresponds to the period 8 orbit and perform a zoom around one of its islands. We calculate the precise initial conditions of this orbit and follow its evolution by increasing the value of the coupling parameter ε as it is shown in fig.3

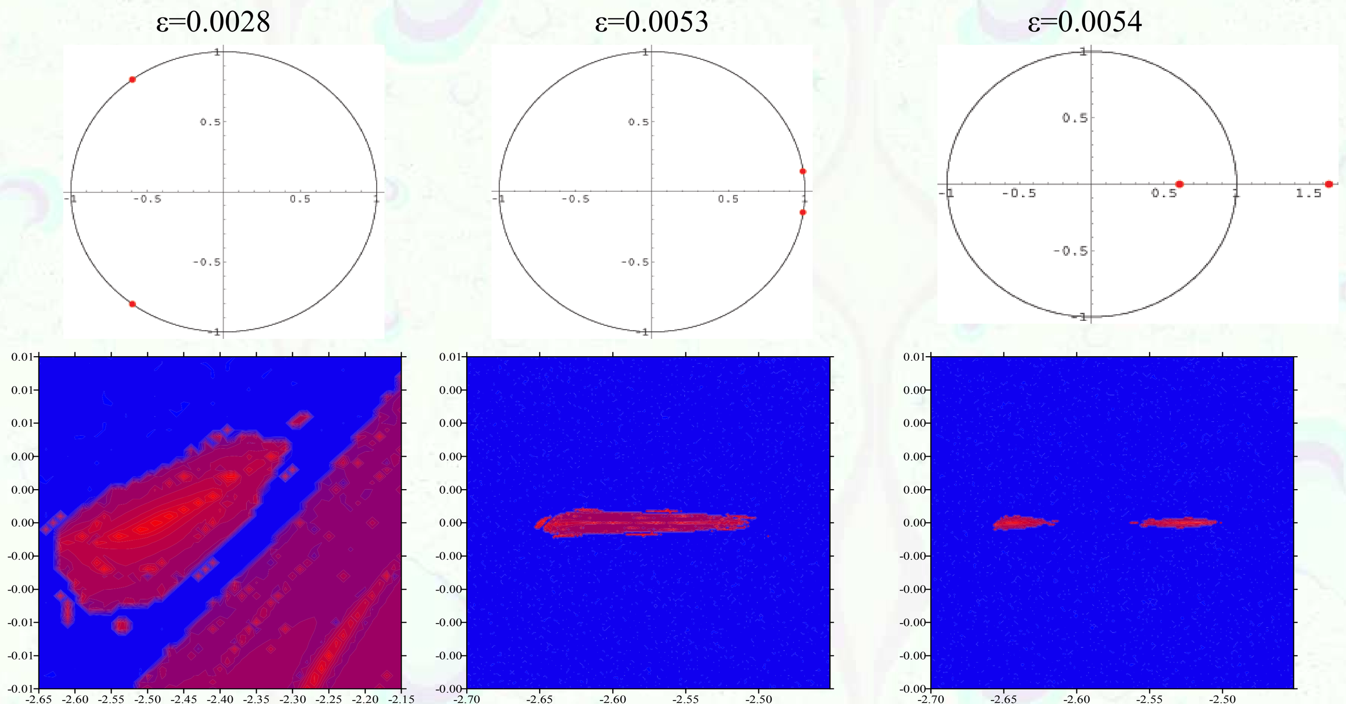

Fig 3: Evolution of the periodic orbit by increasing the perturbation in a 3 oscillators chain.

For ε=0.0028 the eigenvalues of the Jacobian of the variational equations which correspond to the periodic orbit lie in the unit circle, so the orbit is linearly stable. The only case where the eigenvalues is possible to leave the circle is to collide at 1 which takes place for $\varepsilon \simeq 0.0053$ and the orbit destabilizes through a tangent bifurcation which generates two other stable periodic orbits of period 8 each.

After that we consider a 5 oscillator length chain and calculate the precise values of the periodic orbit. We put the oscillators in the initial values of the specific orbit, we scan the initial conditions of the central oscillator and get fig. 4. Since we chose to put the non central oscillators in initial conditions which correspond to the far left island the image we get is obviously asymmetric.

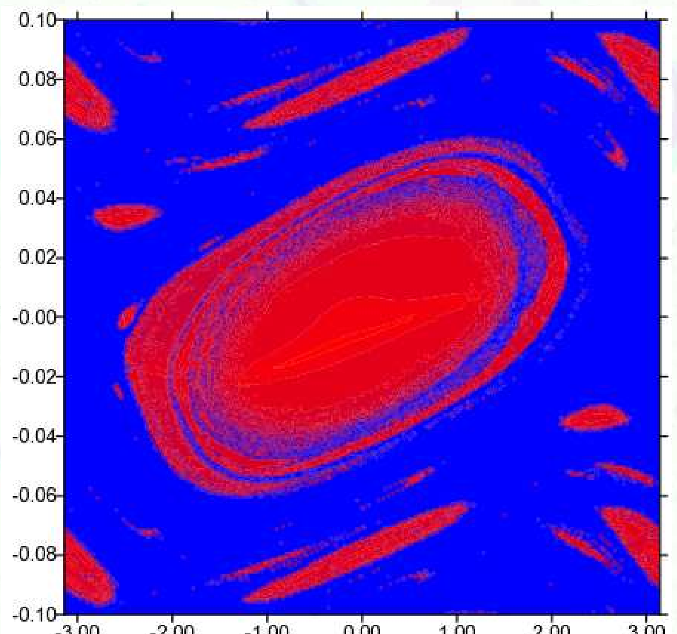

Fig. 4: A full scan of a 5 oscillator chain, putting the non central oscillator in initial conditions which correspond to a period 8 orbit.

We perform a zoom around the specific island and study its evolution with respect to the increasing value of ε.

In this case there are not only the eigenvalues of the central oscillator, so a new scenario of instability is presented. We note that for $\varepsilon \simeq 0.0024$ two pairs of eigenvalues of different Krein kind collide in order to leave the unit circle kai complex instability occurs. (fig. 5)

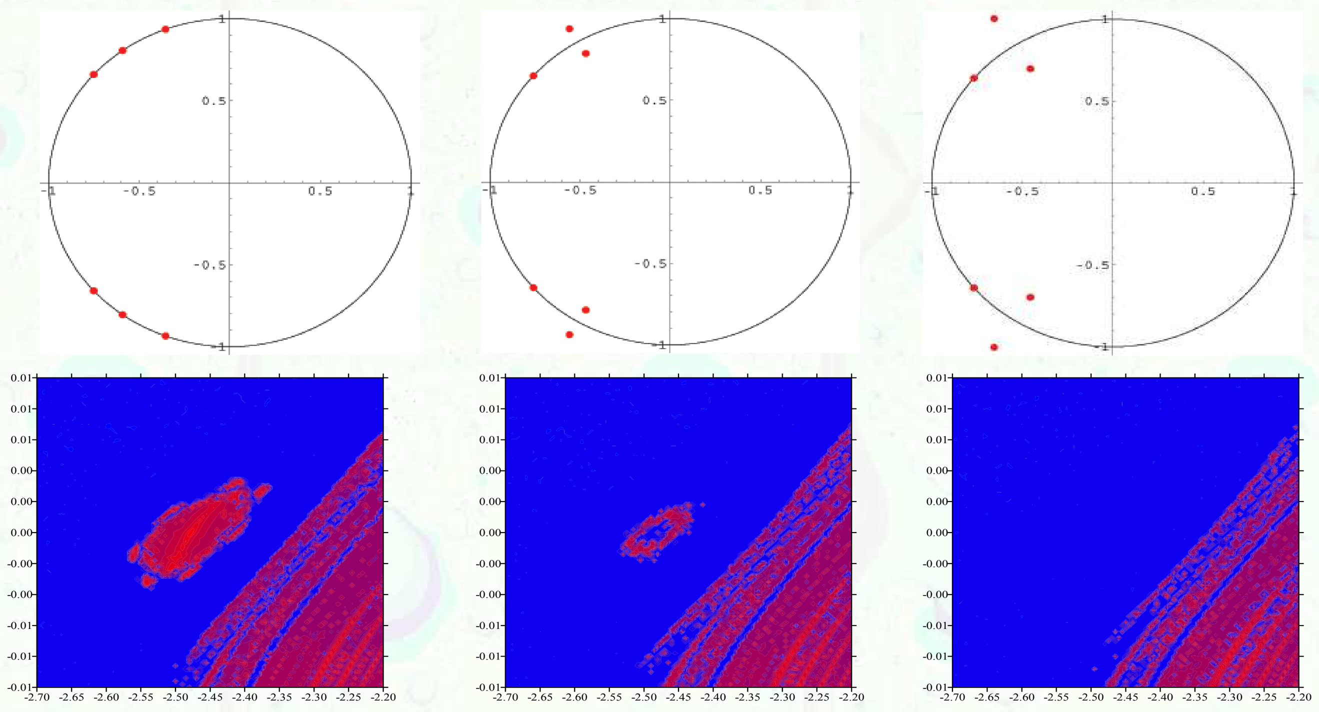

Fig. 5: Evolution of the periodic orbit by increasing the value of ε for a k=5 length chain.

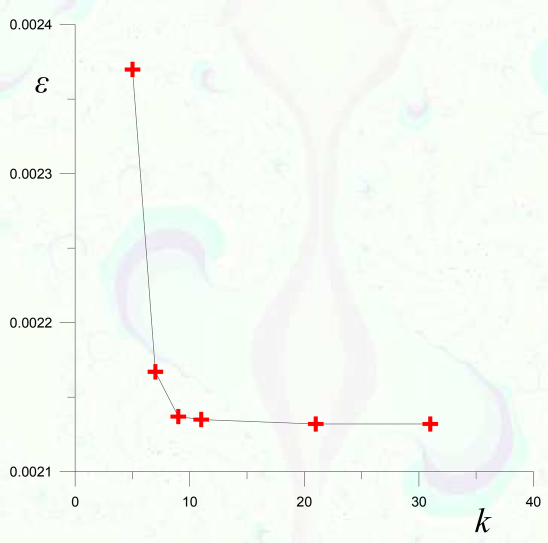

Fig. 6: The value of ε at the bifurcation point versus the length k of the chain

We perform the same analysis of the period 8 orbit stability islands using chain up to length k=21 oscillators. The qualitative behavior of the islands is the same. The only thing that changes is the value of ε in which the bifurcation takes place.

The relation between the length of the chain with the value of the coupling ε in which the bifurcation takes place is shown in fig.6.

We note that the value of ε which is of interest has reached a constant value practically since the value of 9 oscillators. So, we can restrict our study concerning stability matters in chains of 9 or 11 at most oscillators.

IV. Using the scan method in order to locate Discrete Breathers

If we do not know the initial conditions of the periodic orbit which provides the DB, we put all the oscillators at rest (x,y)=(0,0) and scan the allowed region of the initial conditions for the central one. The results of the scan for a 9 oscillator chain are shown in figs. 7 кand 8.

We note that the islands are smaller and not so well formed as before which is expected since we are more distant from the periodic orbit as we were in the previous case. But the qualitative behavior of the diagrams remains the same.

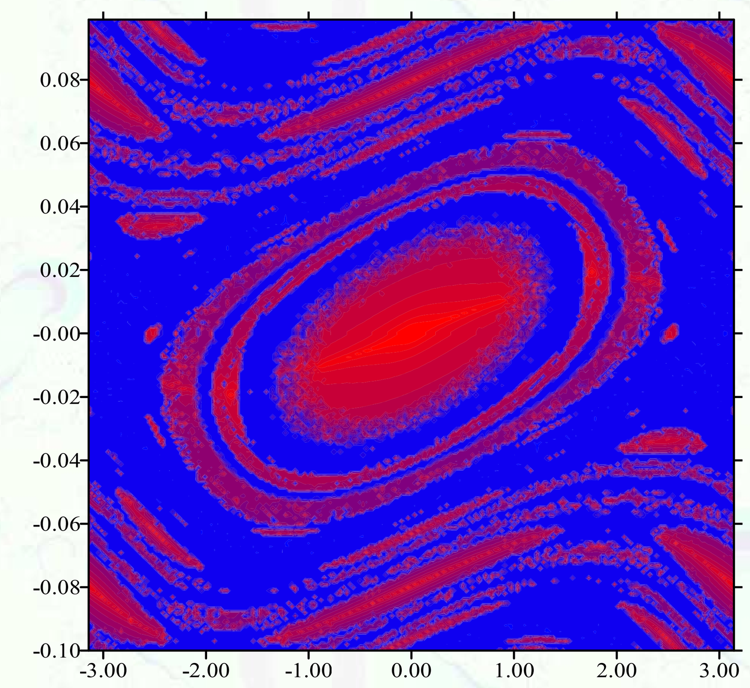

Fig.7 Regular and chaotic orbits in a 5 oscillator chain

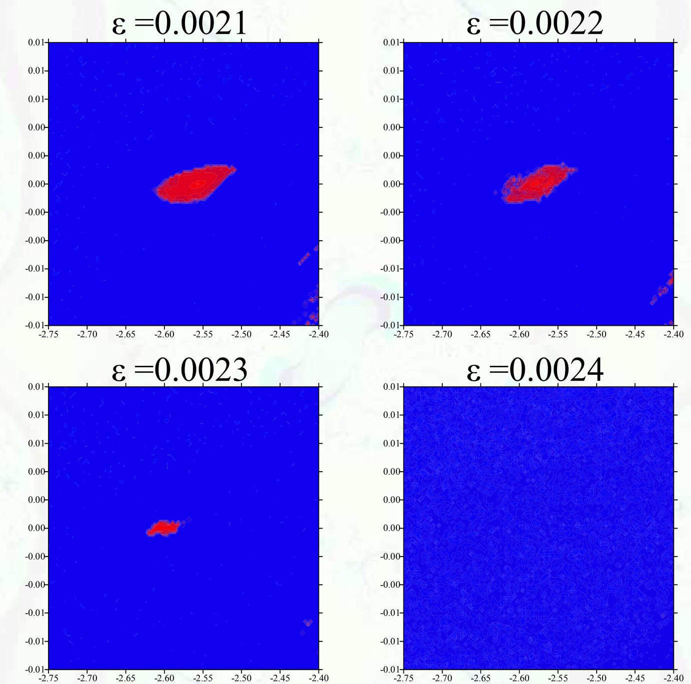

Fig 8: Destruction of a period 8 stability island

We can use an estimate in the region which is shown in fig 8 in order to calculate initial conditions for a Discrete Breather in a chain of 21 oscillators. The resulting breather for ε=0.0021 is shown in fig.9, while, by increasing ε in a value of ε=0.0025 the breather destabilizes (fig.10).

Since we use two dimensional symplectic maps, it is impossible to plot the evolution of both x and y, for all the oscillators of the chain and has no meaning to plot just one of them. Instead we use the integral of the single oscillator $\Phi(x,y)$, which has a direct relation to the amplitude of the oscillation. In figs 9 and 10 we take samples of Φ in multiples of the period (8 in the specific case).

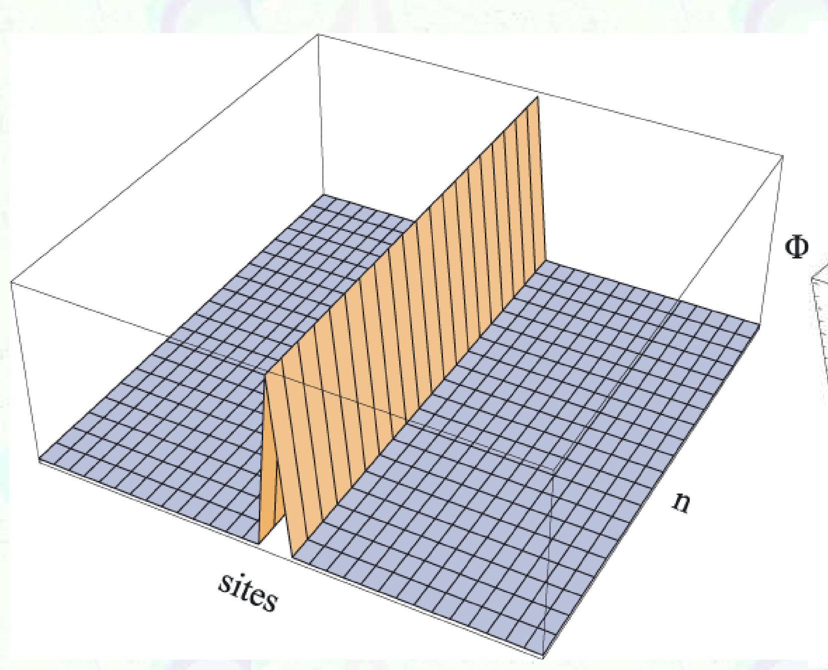

Fig.9: A stable Breather…

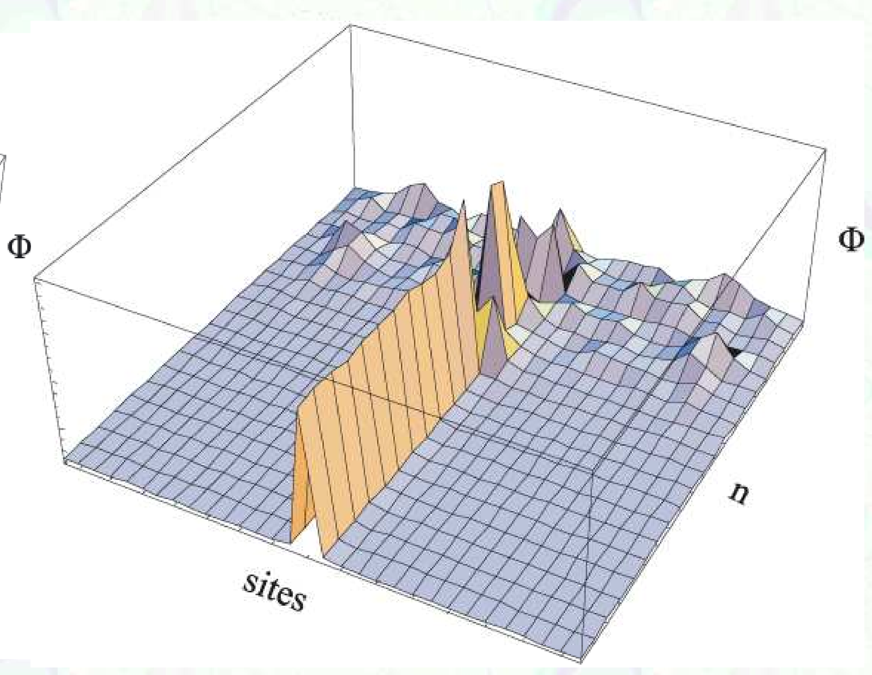

Fig.10: … and its destabilization

IV. Conclusions

The results of paragraph II are in full agreement with the linear theory and we discover that chains of length k=9 are sufficient in order to study the existence and the stability regions of periodic orbits in chains of coupled symplectic maps. In addition, by paragraph III we establish that we can use the scan method which is based on the calculation of FLI in order to locate initial conditions for Discrete Breather and study their stability behavior in correspondence to the variation of the coupling strength ε.

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