



Multisite Discrete Breathers in 1D and 2D Klein-Gordon lattices

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and

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Outline of the talk

- Definition of a Klein-Gordon lattice
- Description of a Discrete Breather
- Existence and Stability results in 1D chains
- Numerical Examples
- Existence and Stability results in 2D lattices
- Numerical Results
- Conclusions

A one-dimensional nonlinear chain



We denote with $V(x_i)$ the **on-site** potential and with $W(x_{i+1} - x_i)$ the **interaction** potential. The general Hamiltonian is of the form

$$H = \sum_i \left(\frac{p_i^2}{2} + V(x_i) \right) + \sum_i W(x_{i+1} - x_i)$$

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Description of an oscillator-Terminology

A nonlinear oscillator is described by a one degree of freedom Hamiltonian, with $V'(0) = 0$ and $V''(0) = \omega_p^2$:

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$$H = H(J) \rightarrow \begin{cases} \dot{w} &= \frac{\partial H}{\partial J} = \omega(J) \\ \dot{J} &= -\frac{\partial H}{\partial w} = 0 \end{cases} \rightarrow \begin{cases} w &= \omega t + \vartheta \\ J &= \text{const.} \end{cases}$$

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Due to the
symmetry:

$$\begin{aligned} x(-t) &= x(t) \\ p(-t) &= -p(t) \end{aligned} \rightarrow x(t) = \sum_{n=0}^{\infty} A_n(J) \cos(nw)$$

A one dimensional chain

Our system is described by a **Klein-Gordon** Hamiltonian

$$H = H_0 + \varepsilon H_1 = \sum_{i=-\infty}^{\infty} \left[\frac{p_i^2}{2} + V(x_i) \right] + \frac{\varepsilon}{2} \sum_{i=-\infty}^{\infty} (x_{i+1} - x_i)^2$$

and the corresponding equations of motion are:

$$\ddot{x}_i = -V'(x_i) - \varepsilon(x_{i+1} - 2x_i + x_{i-1})$$

Each site has **2** neighbors



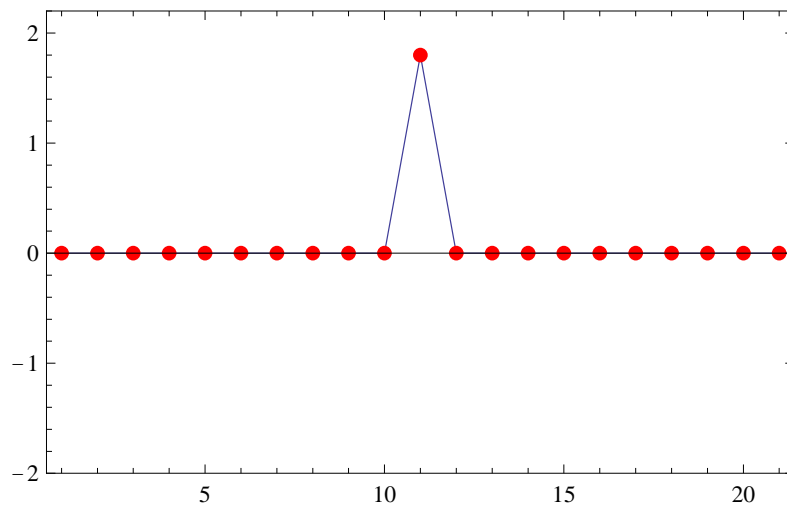
The nearest neighbors are coupled with spring-like forces through ε

The anticontinuous limit

-Consider the anticontinuous or uncoupled limit $\varepsilon = 0$.

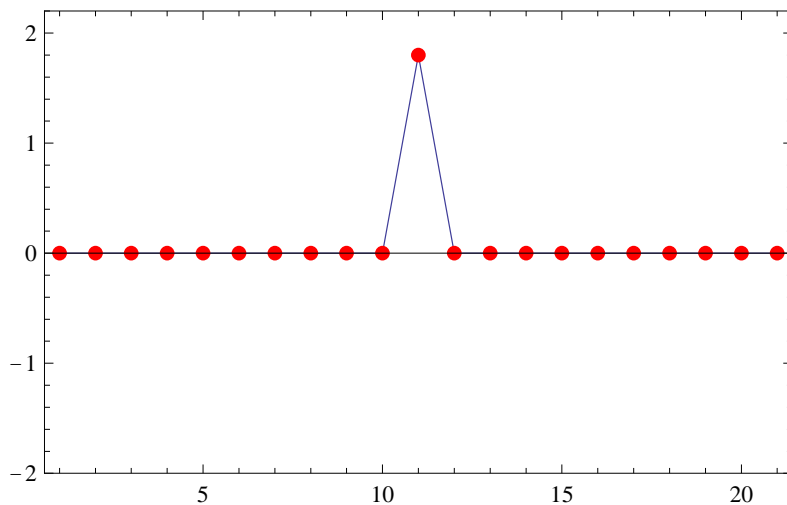
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Is this motion continued for $\varepsilon \neq 0$ to provide a Discrete Breather?

Existence of a Discrete Breather

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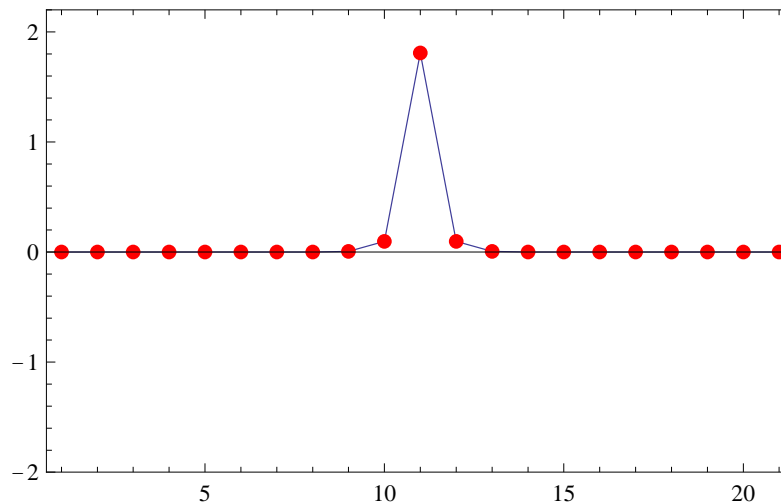
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Linear stability of a breather

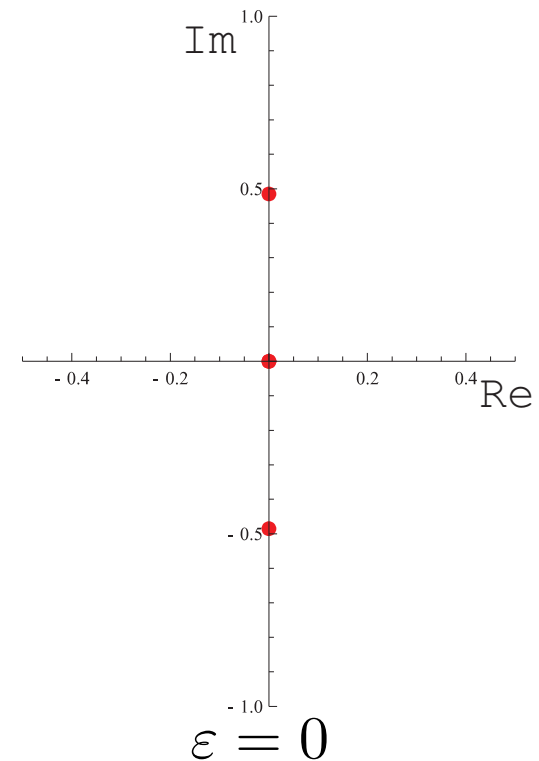
The **characteristic exponents** σ_i of the breather, which are related to the **Floquet multipliers** λ_i of the corresponding periodic orbit as $\lambda_i = e^{\sigma_i T}$. For linear stability **all** σ_i must lie in the imaginary axis.

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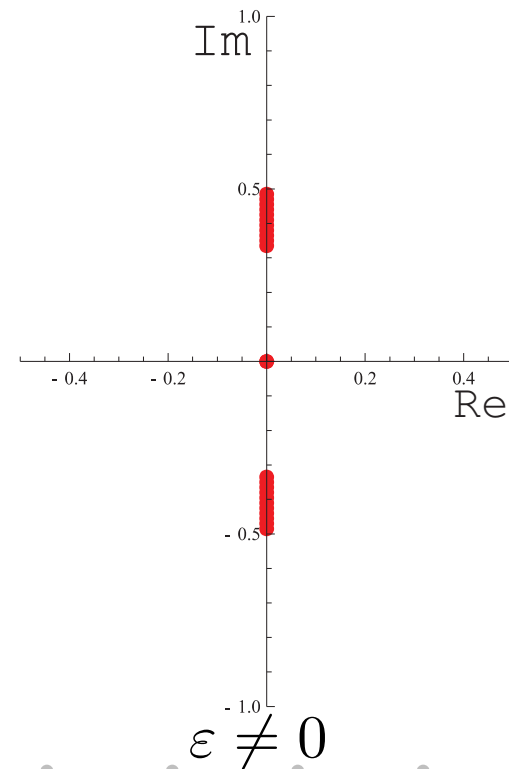
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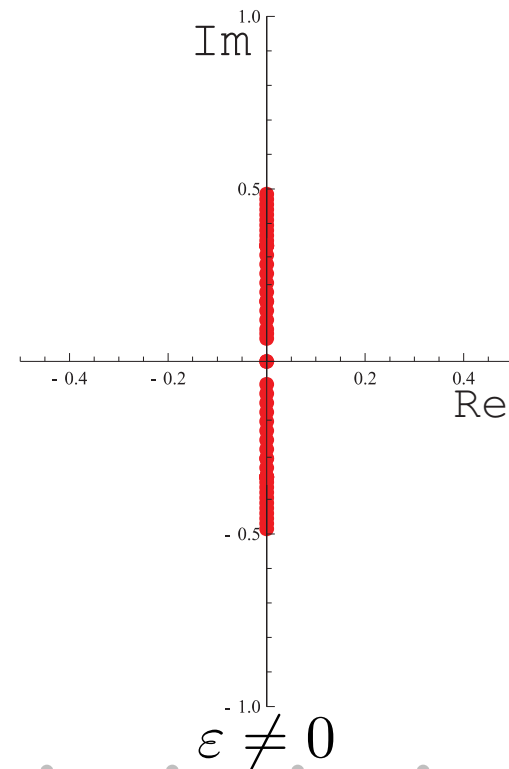
for $\varepsilon \neq 0$
the **phonon band** is generated



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for $\varepsilon \neq 0$
as the value of ε increases
the **phonon band** expands.

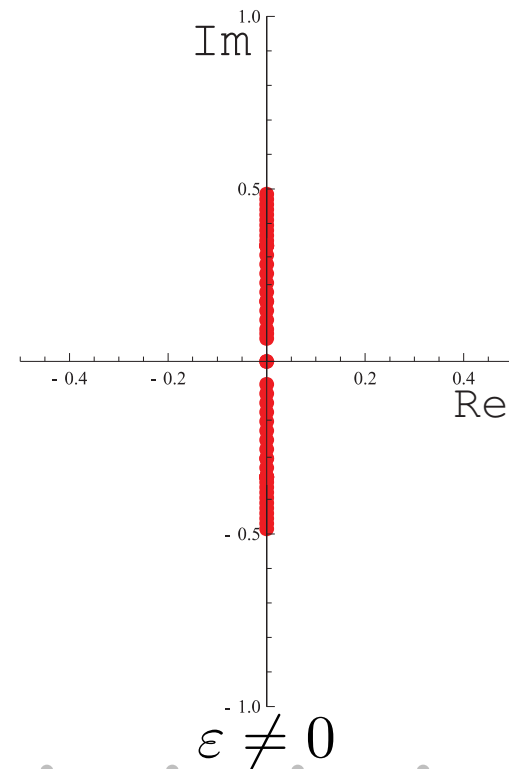


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for $\varepsilon \neq 0$

When $\varepsilon = \varepsilon_{cr}$ the **phonon band** reaches **0** and the breather ceases to exist



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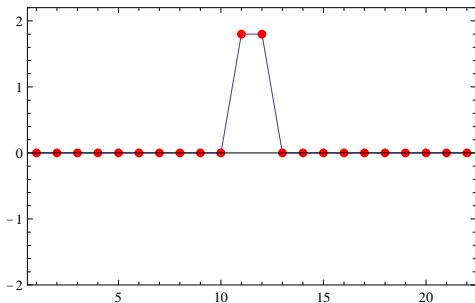
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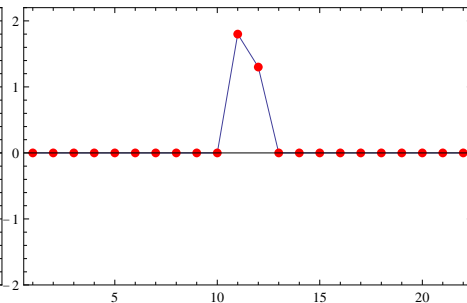
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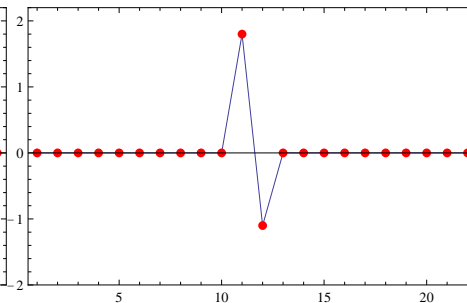
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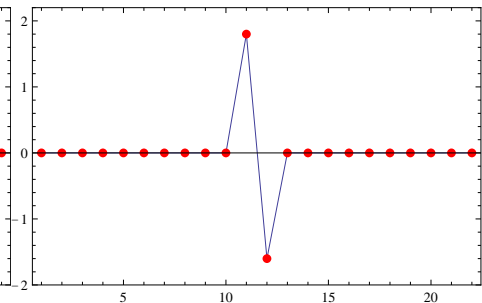
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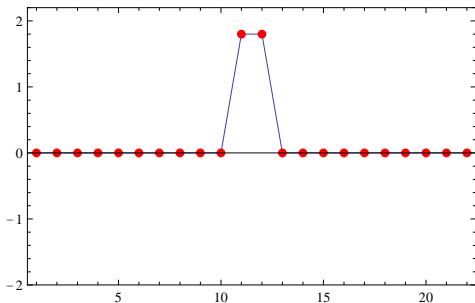
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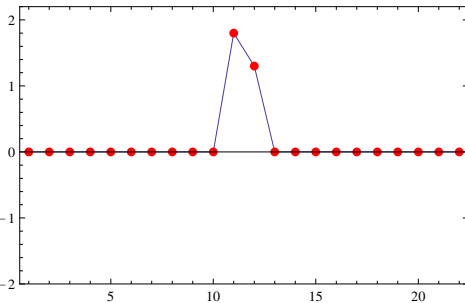
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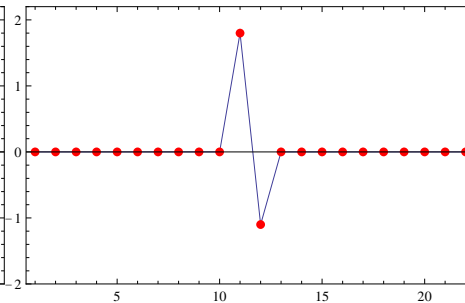
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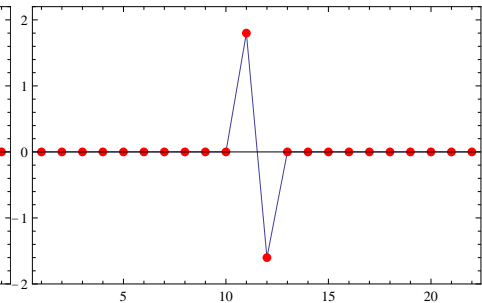
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- Which are the conditions for these motions to be continued as Discrete Multibreathers ?

Existence of multibreathers

Under the nondegeneracy conditions: $\omega_p \neq k\omega$, $\frac{d\omega}{dJ} \neq 0$,
Multibreathers are in one-to-one correspondence with the
singular points of the *Effective Hamiltonian* (R.S. Mackay et al.
(2001,2002), VK and R.S. MacKay(2004))

$$H^{\text{eff}} = \frac{1}{T} \oint H \circ z(t) dt \simeq H_0(I_i) + \varepsilon \langle H_1 \rangle(I_i, \phi_i)$$

where $z(t)$ is the periodic orbit that corresponds to the breather,
while

$$\langle H_1 \rangle = \oint H_1 dt$$

along the **unperturbed** periodic orbit.

Existence of multibreathers

So, the persistence conditions become are:

$$\frac{\partial \langle H_1 \rangle}{\partial \phi_i} = 0, \quad \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_i \phi_j} \neq 0, \quad \omega_p \neq k\omega, \quad \frac{d\omega}{dJ} \neq 0$$

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By using:
$$x(t) = \sum_{m=0}^{\infty} A_m(J) \cos(m\omega)$$

we get:
$$\langle H_1 \rangle = -\frac{1}{2} \sum_{m=1}^{\infty} \sum_{i=1}^n A_m^2 \cos(m\phi_i)$$

and
$$\frac{\partial \langle H_1 \rangle}{\partial \phi_i} = \sum_{m=1}^{\infty} m A_m^2(J) \sin(m\phi_i) = 0$$

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Which is satisfied for any harmonic content if

$$\sin(n\phi_i) = 0 \Rightarrow \phi_i = 0, \pi$$

An open question

Are there any solutions of

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-Again we have seen this numerically, it remains yet to be proved.

Linear stability of a Multibreather

For n – *site breathers* there are n pairs of exponents at zero for $\varepsilon = 0$. These eigenvalues can leave along the imaginary axis (linear stability) or along the real axis (instability).

For example for a 3-breather we can have:

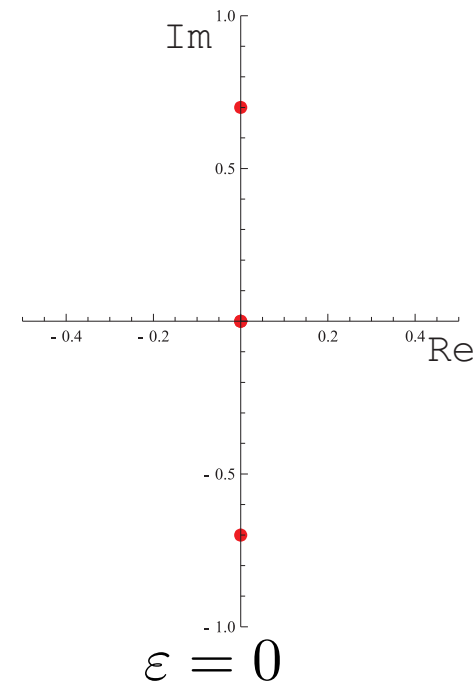
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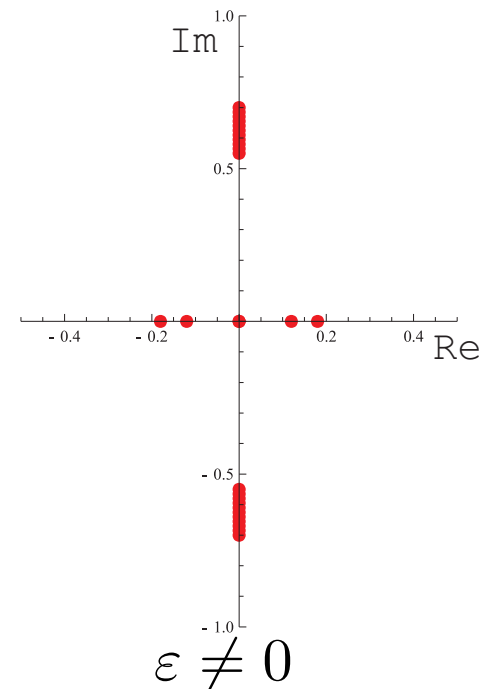
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for $\varepsilon \neq 0$

- the **phonon band** is generated
- **two** pairs of exponents leave along the real axis



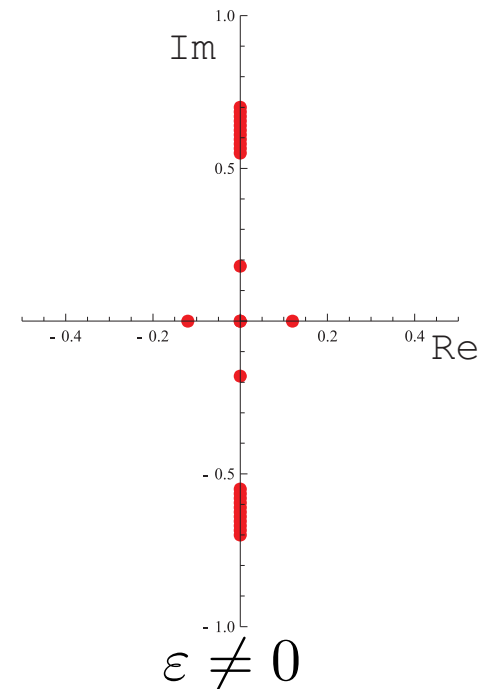
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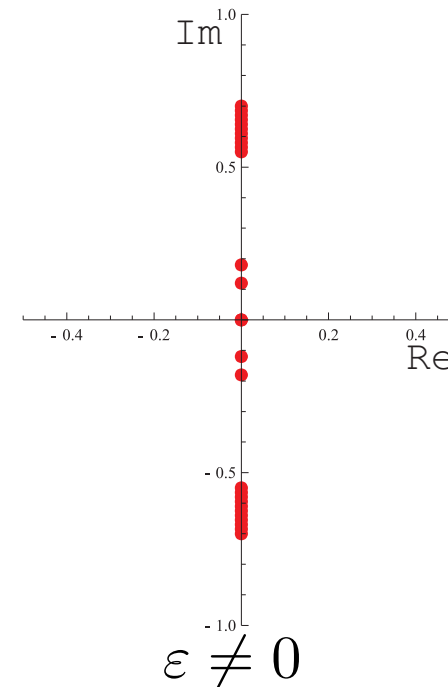
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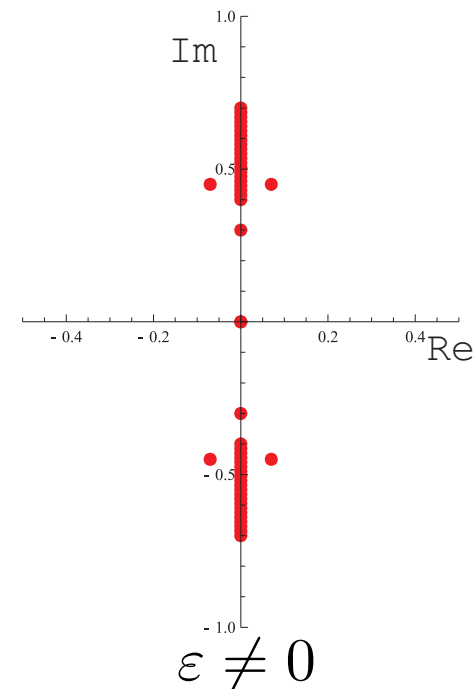
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The configuration remains stable until they collide with the phonon band for

$$\varepsilon = \varepsilon_{cr}$$



A stability Theorem

If $\varepsilon \frac{\partial \omega}{\partial J} < 0$ the only configuration which leads to linearly stable multibreathers, for $|\varepsilon|$ small enough, is the one with $\phi_i = \pi \quad \forall i = 1 \dots n$ (out-of-phase multibreather), while if $\varepsilon \frac{\partial \omega}{\partial J} > 0$ the only linearly stable configuration, for $|\varepsilon|$ small enough, is the one with $\phi_i = 0 \quad \forall i = 1 \dots n$ (in-phase multibreather).

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Moreover, for $\varepsilon \frac{\partial \omega}{\partial J} < 0$ (respectively, $\varepsilon \frac{\partial \omega}{\partial J} > 0$), for unstable configurations, their number of unstable eigenvalues will be precisely equal to the number of nearest neighbors which are in (respectively, out of) phase between them.

VK and P.G. Kevrekidis (2009) arXiv:0902.3990

Outline of the proof

The characteristic exponents of the central oscillators correspond to the eigenvalues of the stability matrix of the fixed points of H^{eff} , $E = \Omega D^2 H^{\text{eff}}$. In leading order of approximation E is

$$E = \begin{pmatrix} \varepsilon \mathbf{A}_1 & \varepsilon \mathbf{B}_1 \\ \mathbf{C}_0 + \varepsilon \mathbf{C}_1 & \mathbf{D}_1 \end{pmatrix} = \left(\begin{array}{c|c} -\varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_i \partial I_j} & -\varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_i \partial \phi_j} \\ \hline \frac{\partial^2 H_0}{\partial I_i \partial I_j} + \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial I_i \partial I_j} & \varepsilon \frac{\partial^2 \langle H_1 \rangle}{\partial \phi_j \partial I_i} \end{array} \right)$$

For linear stability we need **purely imaginary** exponents

Outline of the proof

For the solutions with $\phi_i = 0, \pi$ we have $\mathbf{A}_1 = \mathbf{O}$ and $\mathbf{D}_1 = \mathbf{O}$.
The Stability Matrix E becomes

$$\mathbf{E} = \begin{pmatrix} \mathbf{O} & \varepsilon \mathbf{B}_1 \\ \mathbf{C}_0 + \varepsilon \mathbf{C}_1 & \mathbf{O} \end{pmatrix}$$

The eigenvalues σ_i of E are of the form $\sigma_i = \pm\sqrt{\varepsilon\chi_i} + \mathcal{O}(\varepsilon^{3/2})$
where χ_i are the eigenvalues of $\mathbf{B}_1\mathbf{C}_0$.

So, the eigenvalue problem of E becomes an eigenvalue
problem of $\mathbf{B}_1\mathbf{C}_0$

Outline of the proof

$$\mathbf{B}_1 \cdot \mathbf{C}_0 = -\frac{\partial \omega}{\partial J} \mathbf{Z} = -\frac{\partial \omega}{\partial J} \begin{pmatrix} 2f_1 & -f_1 & 0 & & \\ -f_2 & 2f_2 & -f_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -f_{n-1} & 2f_{n-1} & -f_{n-1} \\ & & 0 & -f_n & 2f_n \end{pmatrix}$$

$$\sigma_{\pm i} = \pm \sqrt{-\varepsilon \frac{\partial \omega}{\partial J} z_i} + \mathcal{O}(\varepsilon^{3/2}) \text{ and } f_i = f(\phi_i) = \sum_{n=1}^{\infty} n^2 A_n^2 \cos(n\phi_i)$$

-Assuming the absence of solutions other than $\phi_i = 0, \pi$, then $f(0) > 0$ and $f(\pi) < 0$.

-Then, the number of positive eigenvalues z_i 's equals the number of positive f_i 's, while the number of negative z_i 's equals the number of negative f_i 's.

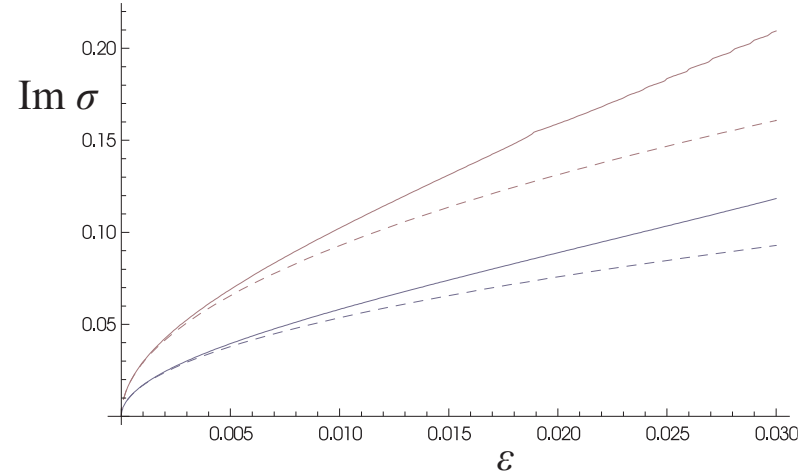
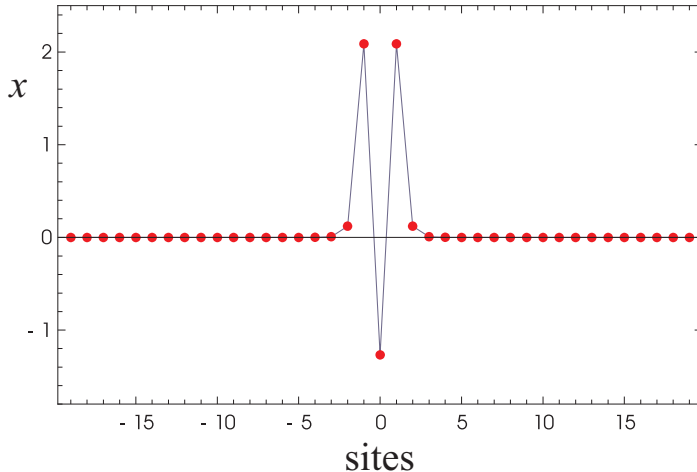
Outline of the proof

- If $\varepsilon \frac{\partial \omega}{\partial J} < 0$ and $\phi_i = \pi \quad \forall i = 1 \dots n$, or if $\varepsilon \frac{\partial \omega}{\partial J} > 0$ and $\phi_i = 0 \quad \forall i = 1 \dots n$, then all the eigenvalues of E are purely imaginary up to $\mathcal{O}(\sqrt{\varepsilon})$ terms.
- If the eigenvalues of \mathbb{E} are imaginary up to $\mathcal{O}(\sqrt{\varepsilon})$ terms they remain imaginary up to all orders of approximation. This is due to symplectic signature reasons.

An out of phase 3-site breather

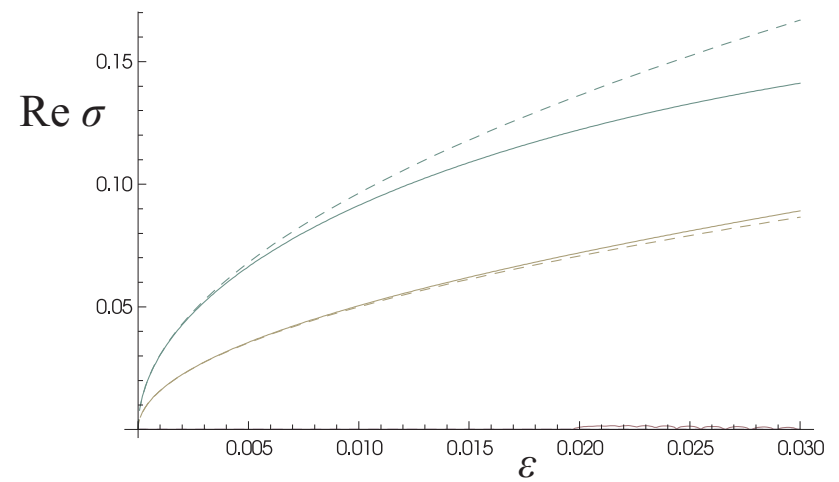
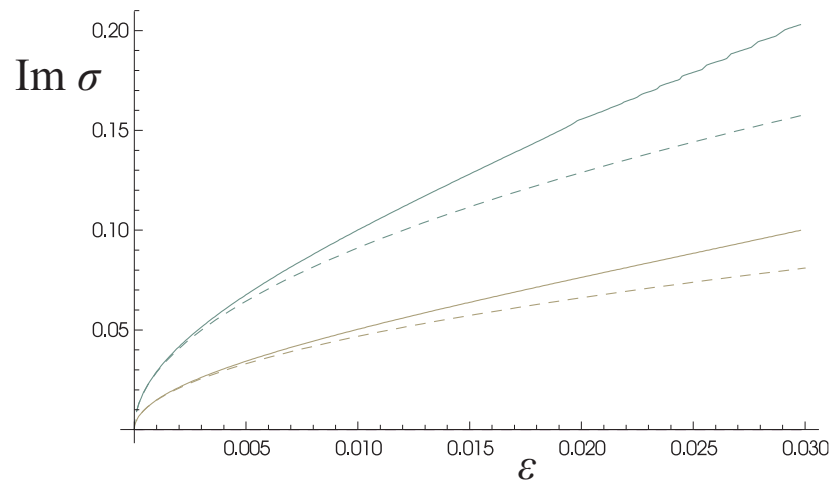
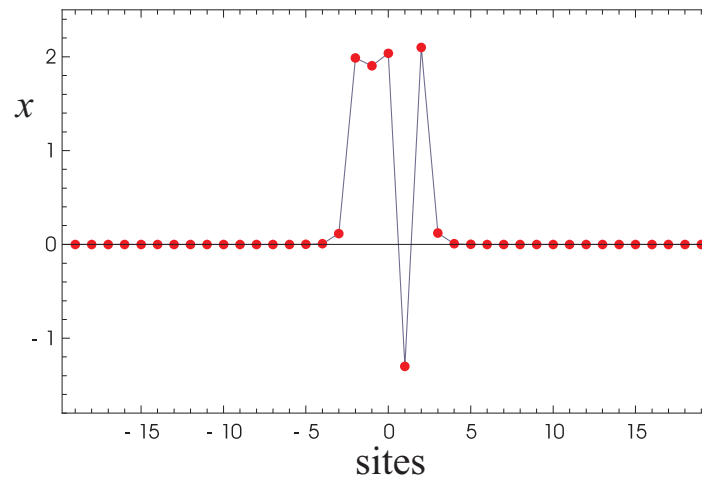
In the case of a 3-site breather we have just 2 positive σ_i

$$\sigma_{\pm 1} = \pm \sqrt{-\varepsilon \frac{\partial \omega}{\partial J} f(\pi)} \quad , \quad \sigma_{\pm 2} = \pm \sqrt{-3\varepsilon \frac{\partial \omega}{\partial J} f(\pi)}.$$



A 5-site breather

In this case we have 4 ϕ_i : $\phi_{1,2} = 0$ and $\phi_{3,4} = \pi$.



Sumarizing...

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- We categorize the multibreathers in terms of their stability, by knowing the sign of $\varepsilon \frac{\partial \omega}{\partial J}$ and the values of ϕ_i .

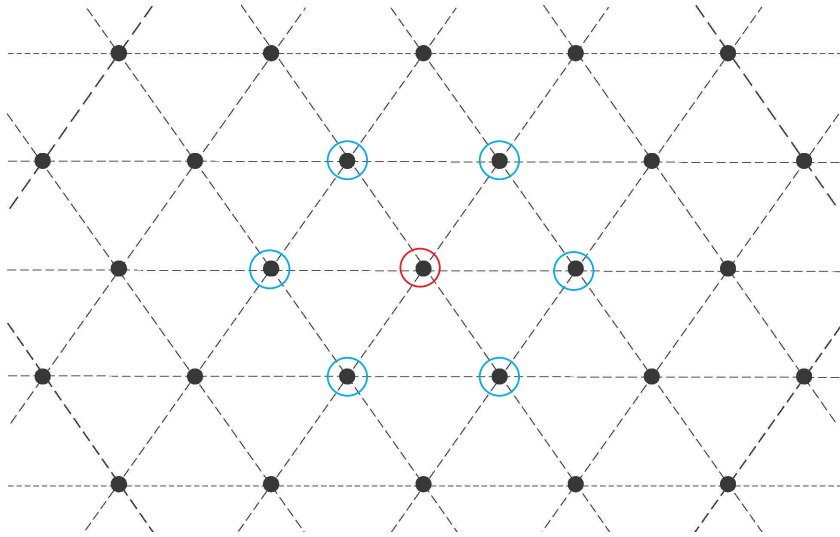
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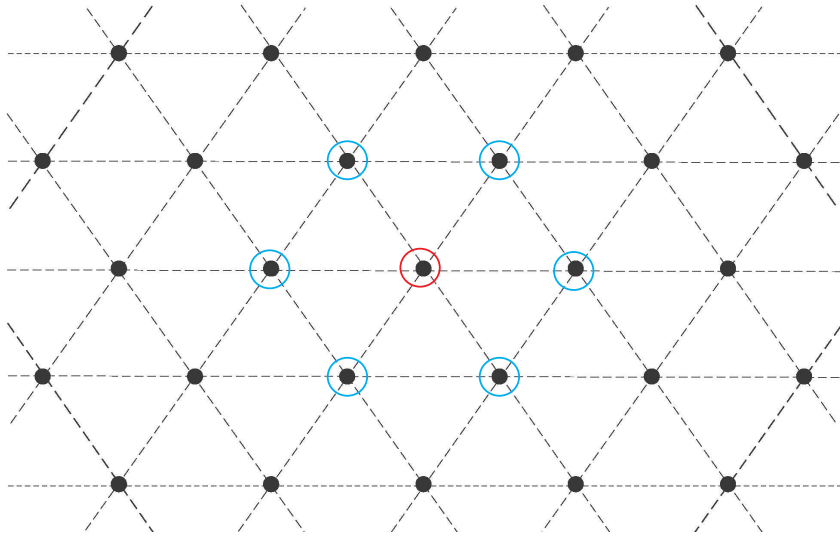
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The Hexagonal Lattice



Each site has 6 neighbors

The Hexagonal Lattice



Each site has **6** neighbors

The sites are coupled through ε

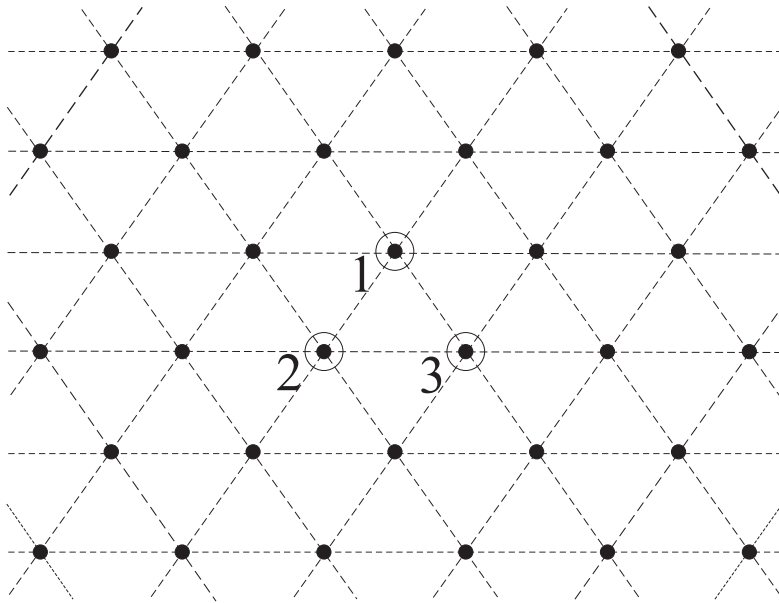
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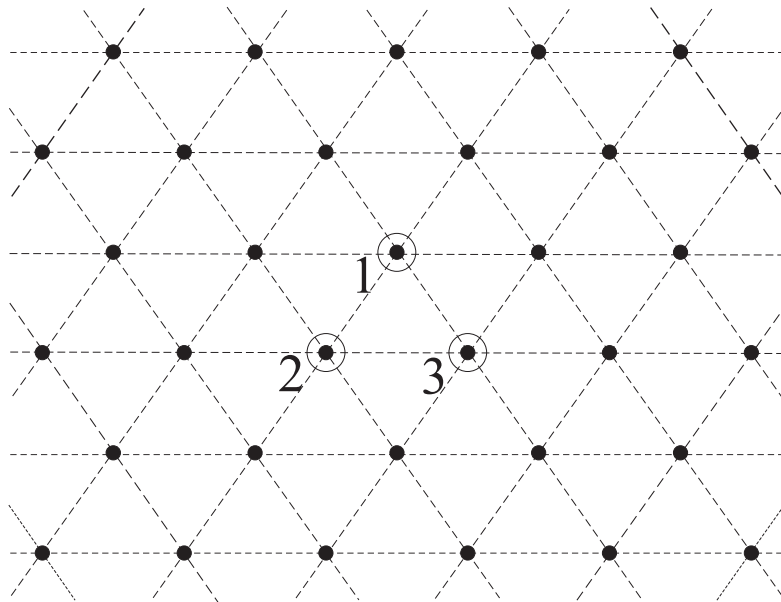
$$\ddot{x}_i = -V'(x_i) - \varepsilon \left(\sum_{j \in \mathbb{G}_i} x_j - \mathbf{6}x_i \right)$$

Hexagonal Lattice-Anticontinuous Limit



In the anticontinuous limit $\varepsilon = 0$ we consider 3 oscillators moving in periodic orbits with frequency ω .

Hexagonal Lattice-Anticontinuous Limit



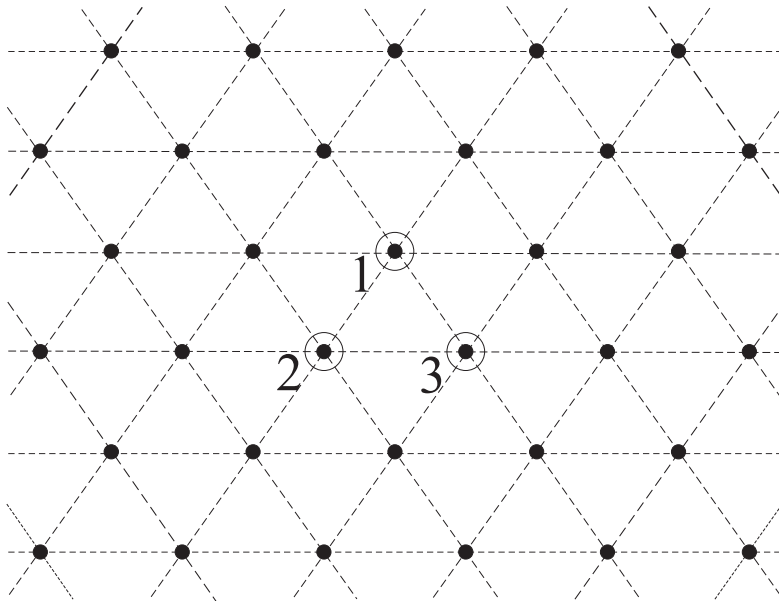
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Only 2 independent ϕ_i : $\phi_1 = w_2 - w_1, \phi_2 = w_3 - w_2$

Persistence condition for 3-site breathers

Using the previous terminology, the persistence condition becomes

$$\sin(n\phi_i) + \sin[n(\phi_1 + \phi_2)] = 0, \quad i = 1, 2 \quad n \in \mathbb{N}$$

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$\phi_{1,2} = 0 \quad \rightarrow \quad \text{in-phase tree-site breather}$

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$\phi_{1,2} = \frac{2\pi}{3} \quad \rightarrow \quad \text{vortex breather}$

Stability in the hexagonal lattice

Defining:
$$f(\phi) = \frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \cos(n\phi)$$

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$$\phi_i = \pi \quad \rightarrow \quad \sigma_i = \begin{cases} \pm \sqrt{-\varepsilon \frac{d\omega}{dJ} (2f(0) + f(\pi))} \\ \pm \sqrt{-\varepsilon \frac{d\omega}{dJ} f(\pi)} \end{cases}$$

$$\phi_i = 2\pi/3 \quad \rightarrow \quad \sigma_i = \pm \sqrt{-3\varepsilon \frac{d\omega}{dJ} f(2\pi/3)}$$

Stability for a specific example

For: $\varepsilon > 0$ and $V(x) = \frac{x^2}{2} - 0.27\frac{x^3}{3} - 0.03\frac{x^4}{4} \rightarrow \frac{d\omega}{dJ} < 0$

Stability for a specific example

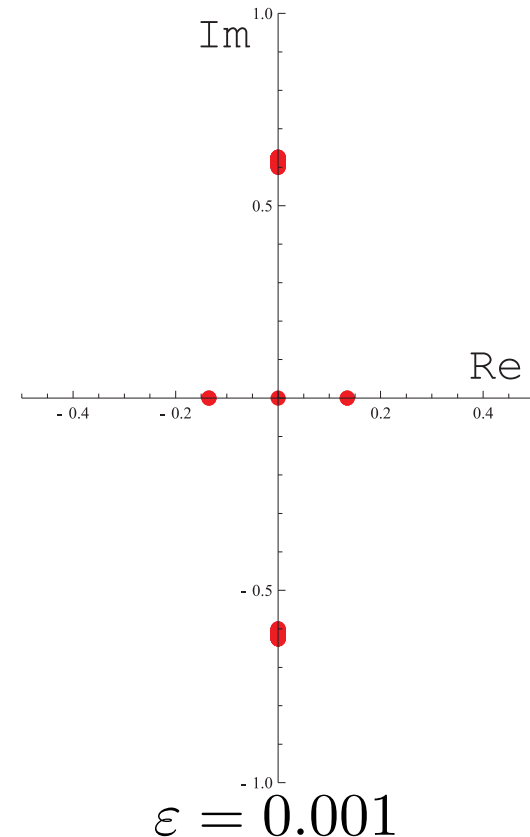
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In phase : $\phi_i = 0$

$$f(0) = \sum_{n=1}^{\infty} n^2 A_n^2 > 0$$

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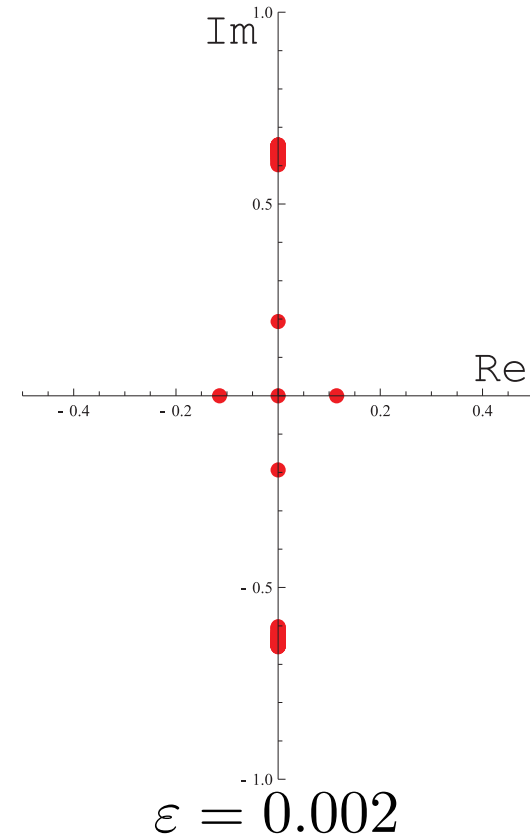
$$f(\pi) = \sum_{n=1}^{\infty} (-1)^n n^2 A_n^2 < 0$$

$$2f(0) + f(\pi) > 0$$

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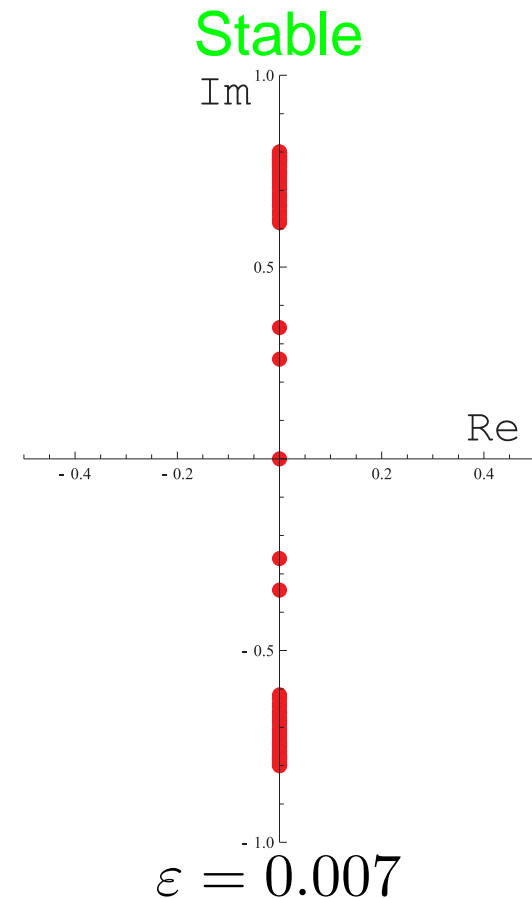
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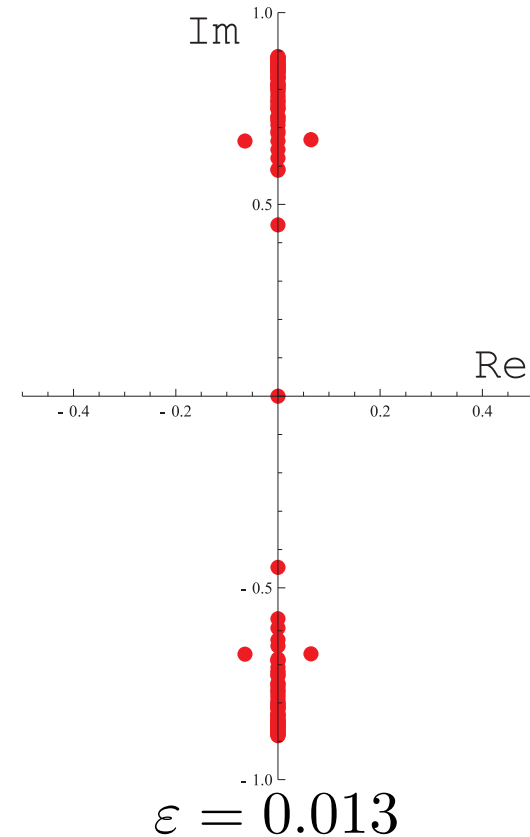
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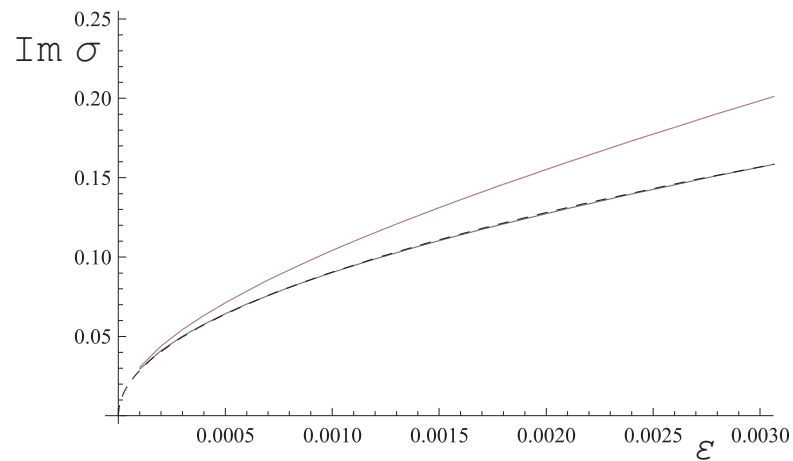
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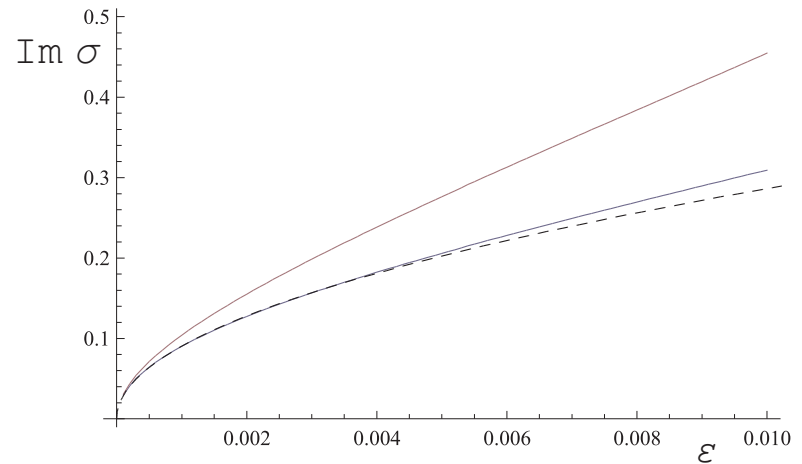
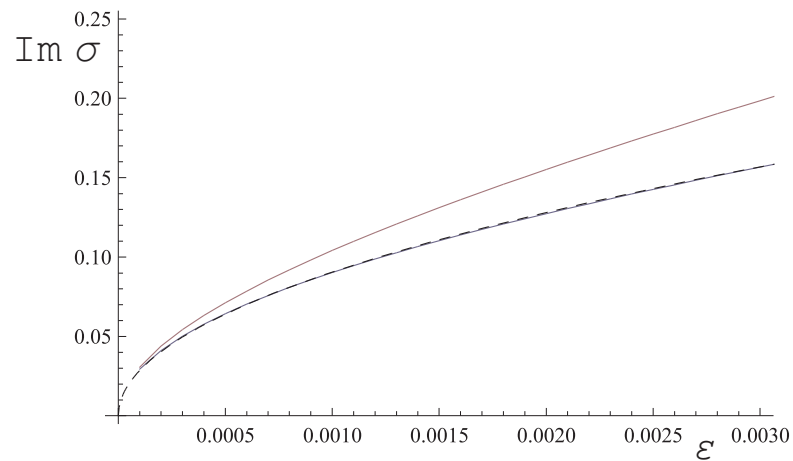
Prediction vs Numerics

We draw our $O(\sqrt{\varepsilon})$ prediction together with the numerical results

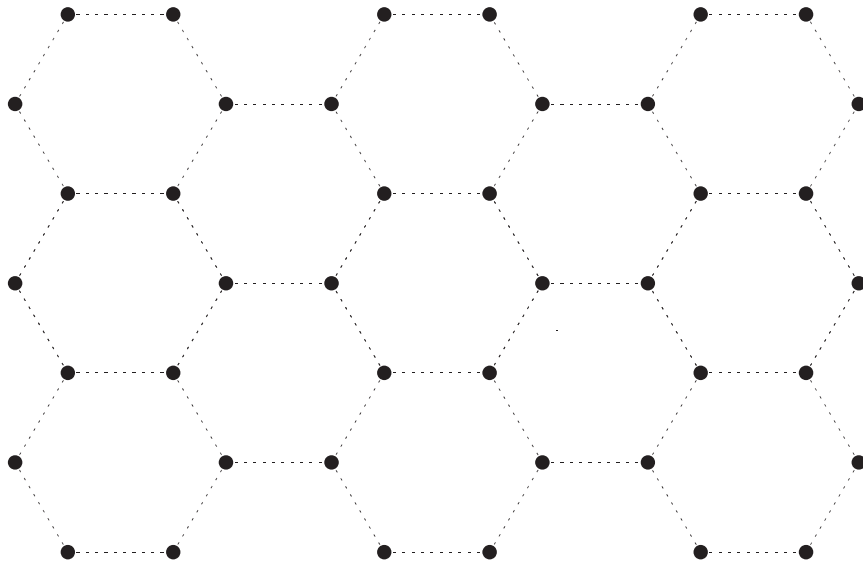


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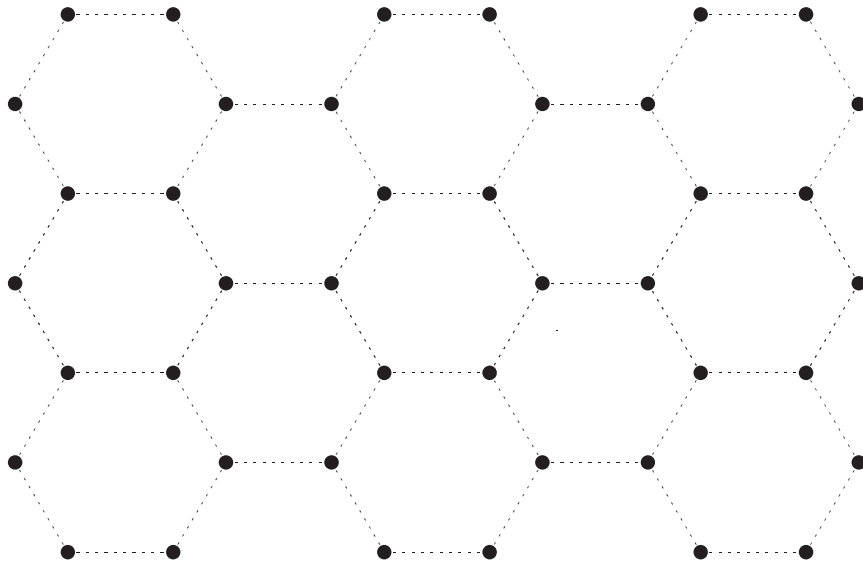


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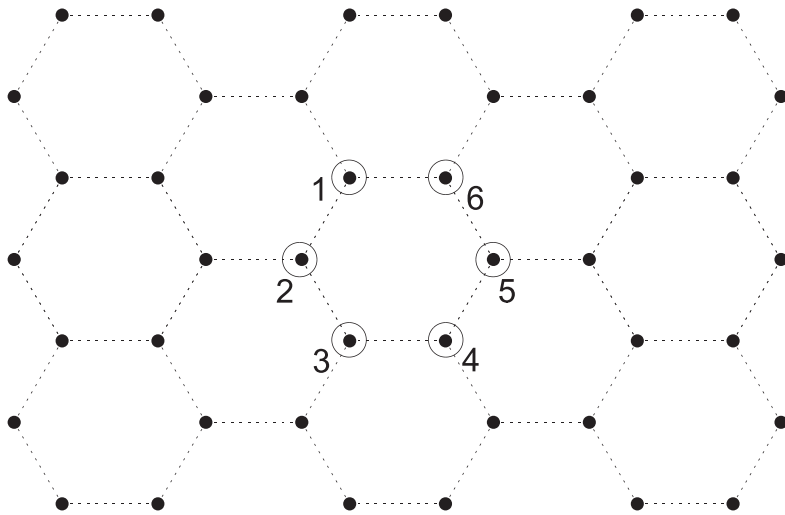
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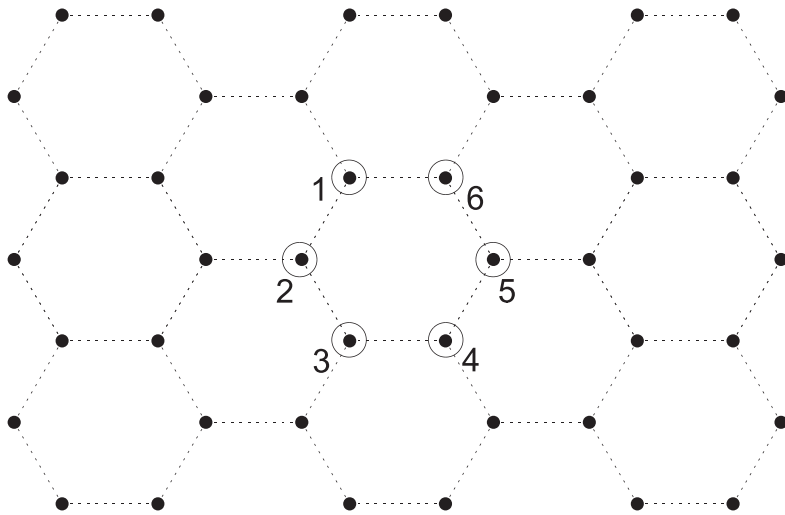
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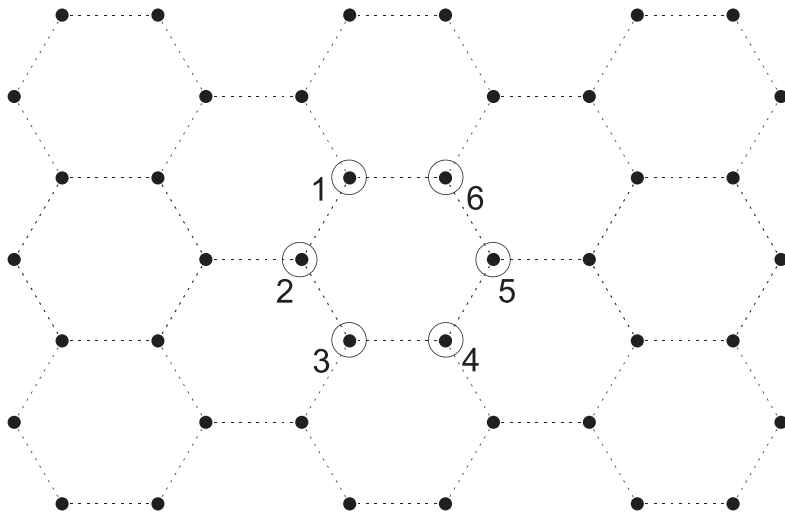
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5 independent ϕ_i : $\phi_1 = w_2 - w_1$, $\phi_2 = w_3 - w_2$, $\phi_3 = w_4 - w_3$
 $\phi_4 = w_5 - w_4$, $\phi_5 = w_6 - w_5$

Persistence conditions for 6-site breathers

The persistence conditions in this case become

$$\sin(n\phi_i) + \sin[n(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5)] = 0, i \in 1..5, n \in \mathbb{N}$$

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Two groups of solutions:

The **6-site breathers**

$$\phi_i = 0, \pi \quad i = 1..5$$

The **vortex** breathers

$$\phi_i = s \frac{\pi}{3} \quad i = 1..5 \quad s \in \mathbb{N}$$

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$$\phi_i = 0 \quad \rightarrow \quad \text{in phase 6-breather}$$

$$\phi_i = \pi \quad \rightarrow \quad \text{out of phase 6-breather}$$

$$\phi_i = \frac{\pi}{3} \quad \rightarrow \quad \text{charge 1 vortex breather}$$

$$\phi_i = \frac{2\pi}{3} \quad \rightarrow \quad \text{charge 2 vortex breather}$$

Stability in the Honeycomb lattice

Using again:

$$f(\phi) = \frac{1}{2} \sum_{n=1}^{\infty} n^2 A_n^2 \cos(n\phi)$$

$$\sigma_i = \begin{cases} \pm \sqrt{-\varepsilon \frac{d\omega}{dJ} f(\phi)} & i \in 1..4 \\ \pm \sqrt{3} \sqrt{-\varepsilon \frac{d\omega}{dJ} f(\phi)} & i \in 5..8 \\ \pm 2 \sqrt{-\varepsilon \frac{d\omega}{dJ} f(\phi)} & i \in 9, 10 \end{cases}$$

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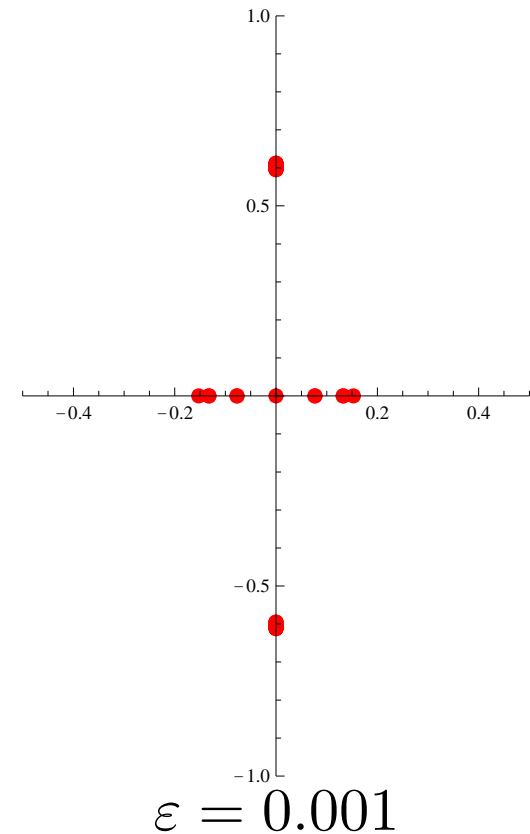
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$$f(0) = \sum_{n=1}^{\infty} n^2 A_n^2 > 0$$

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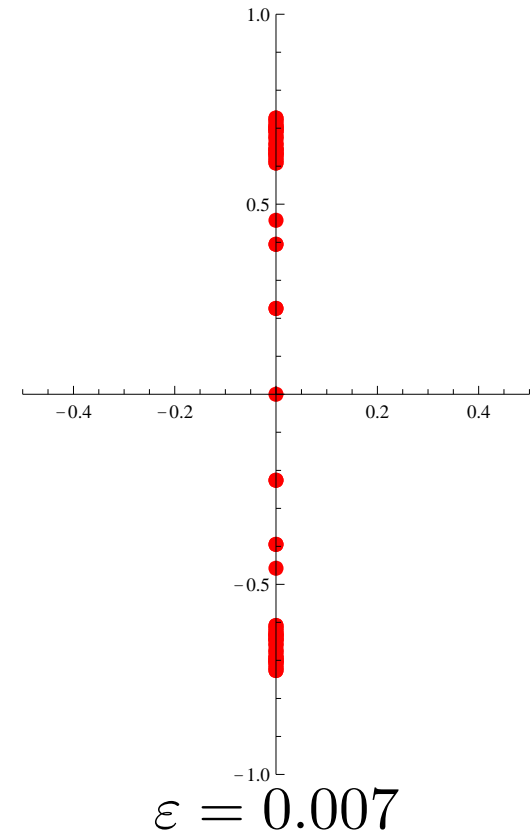
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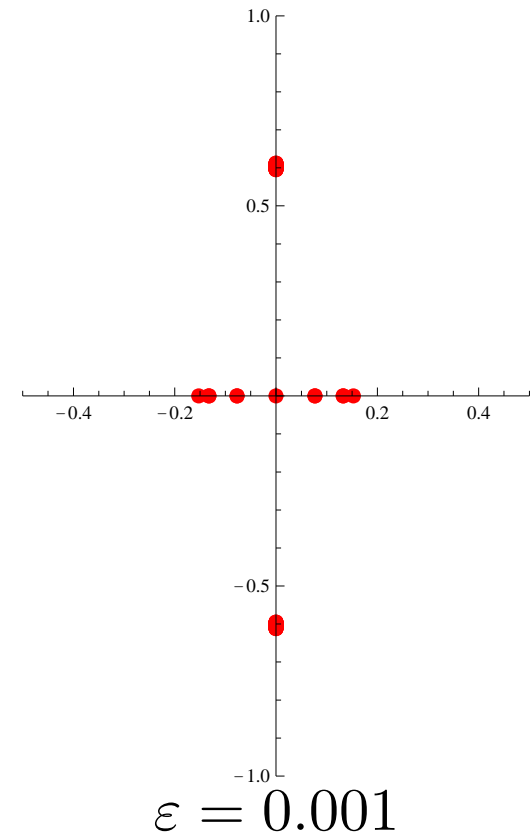
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Charge 1 Vortex : $\phi_i = \pi/3$

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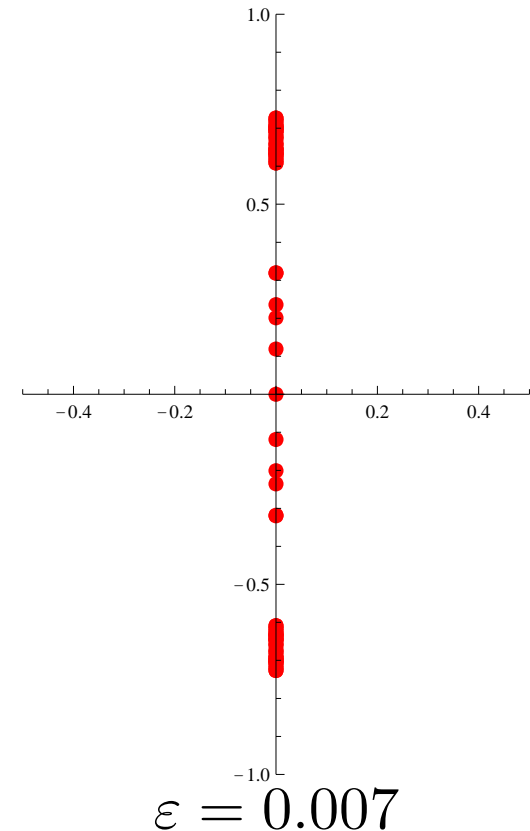
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Charge 2 Vortex : $\phi_i = 2\pi/3$

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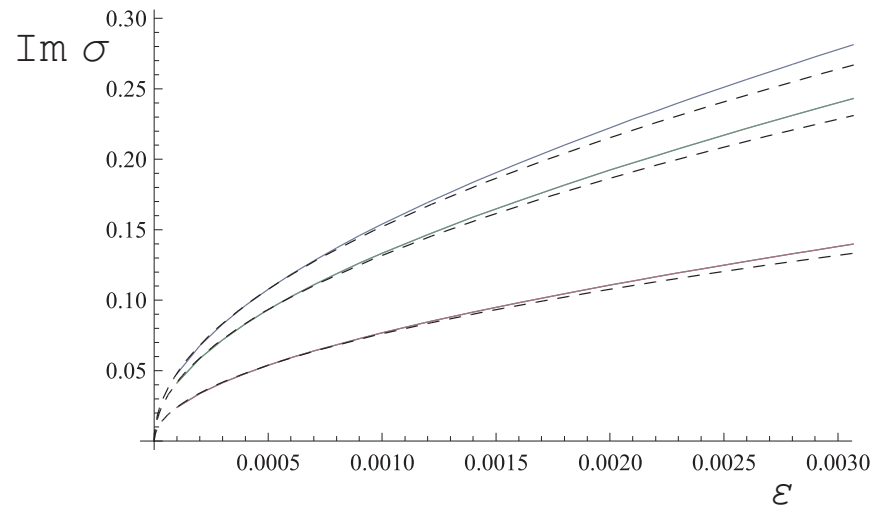
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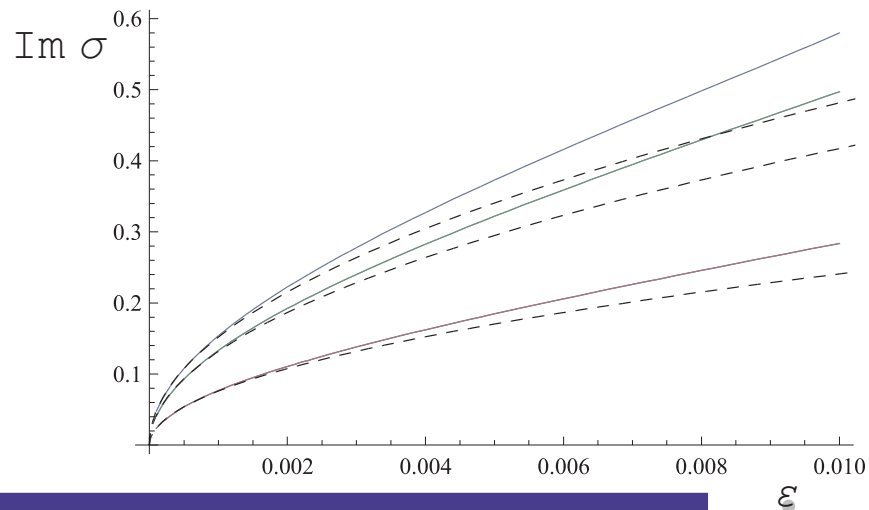
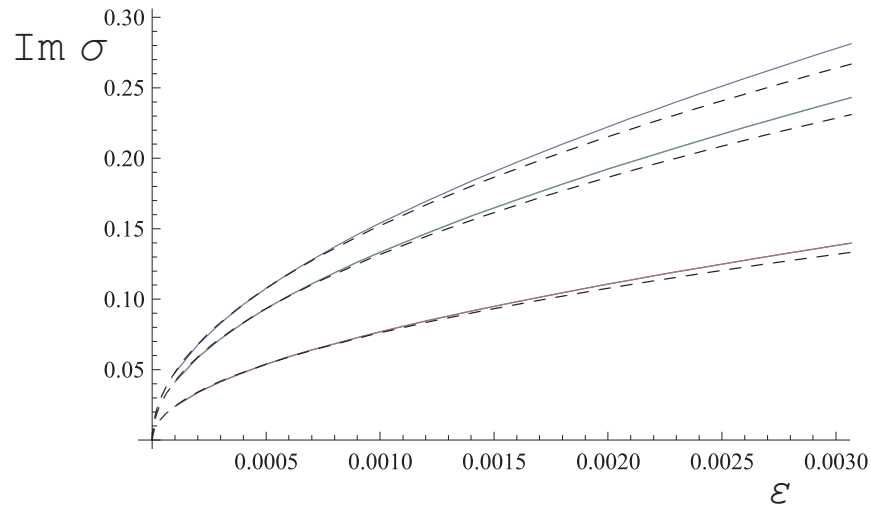
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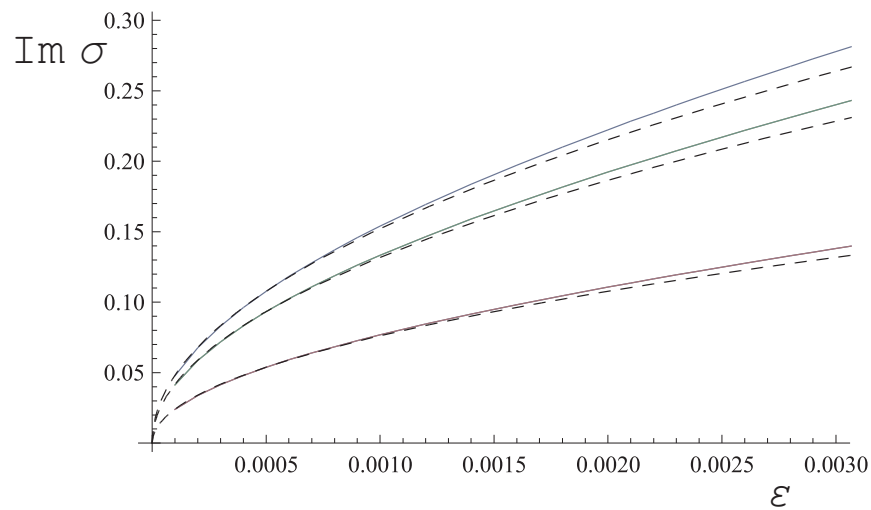
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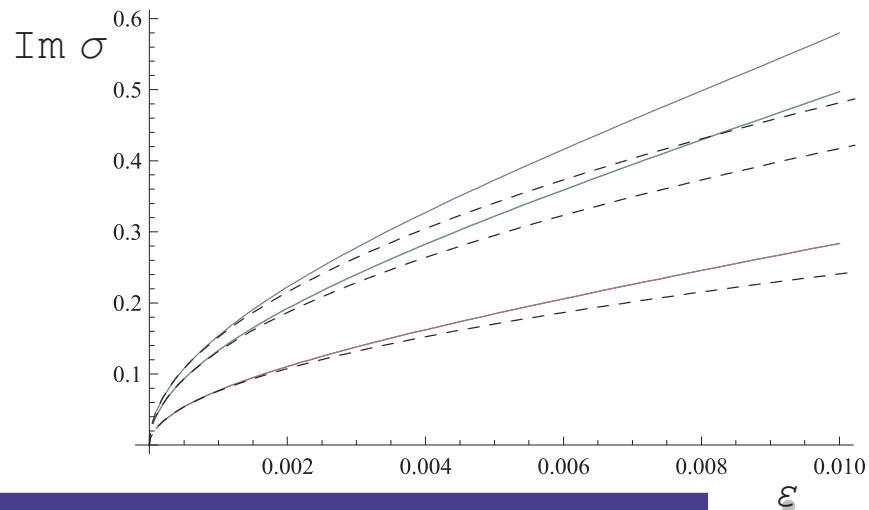
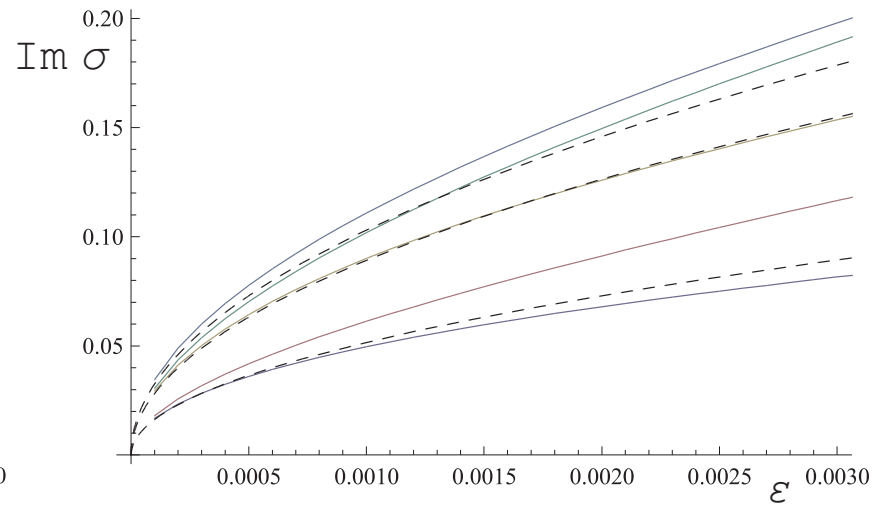


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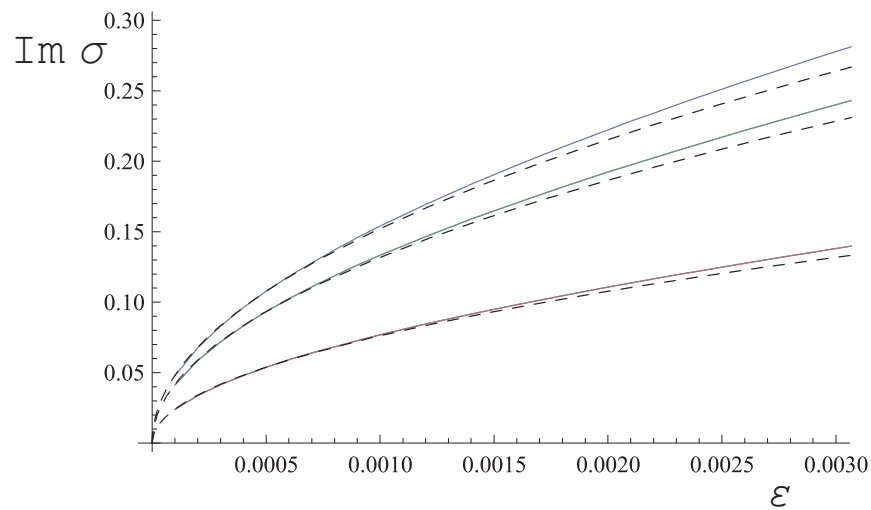


Vortex $s = 2$

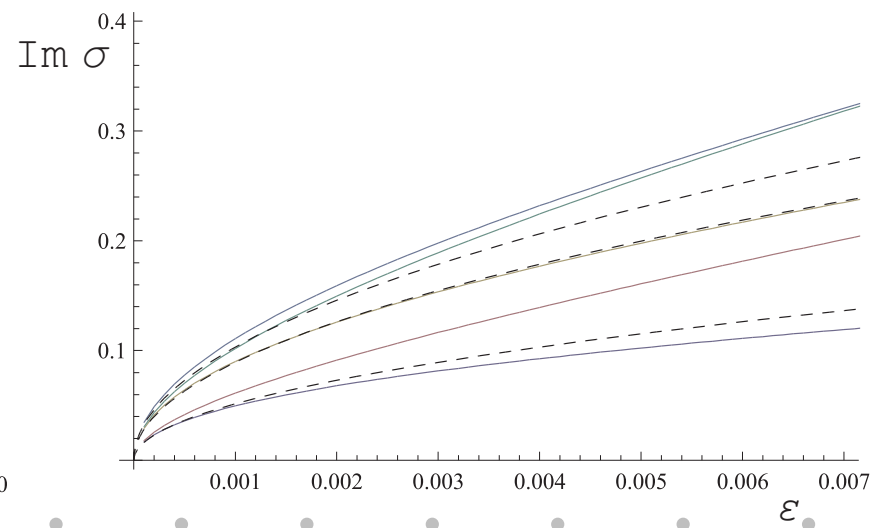
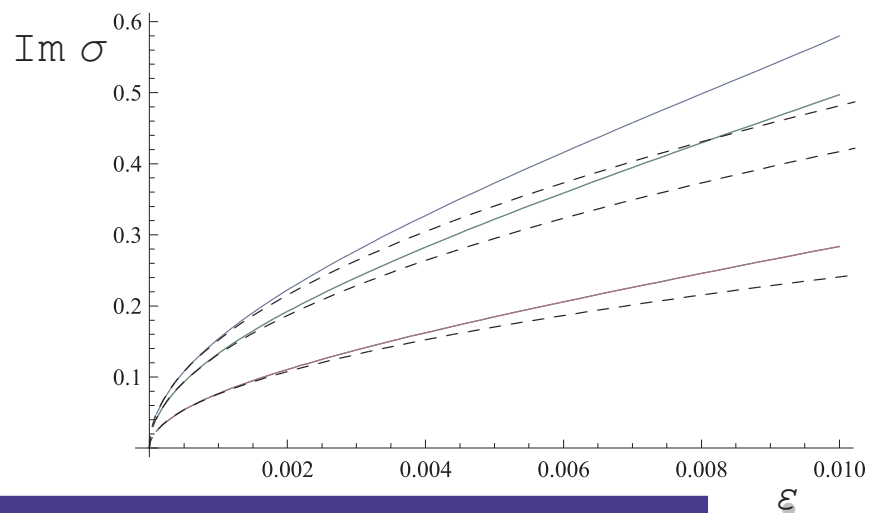
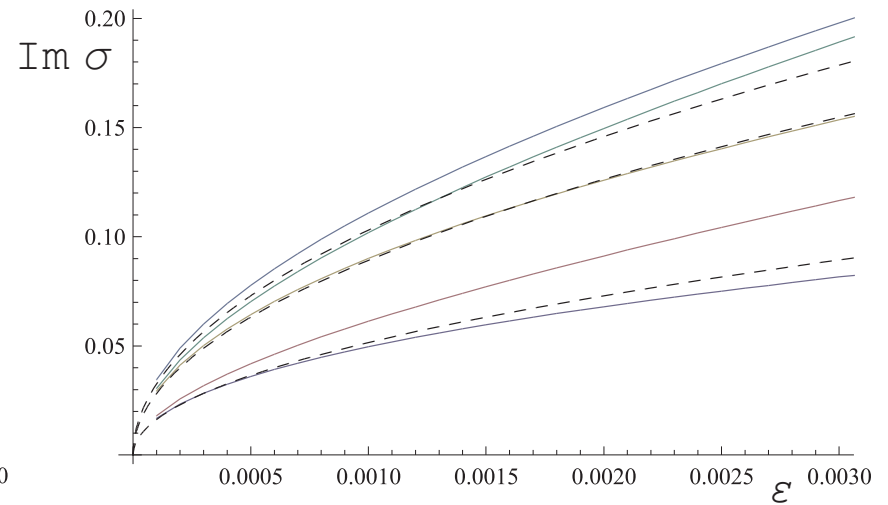


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- We categorize these solutions in terms of their linear stability.
- We calculate an $\mathcal{O}(\sqrt{\varepsilon})$ estimate of the corresponding exponents.
- We cannot have larger rings. A higher order theory would be useful.