Existence and Stability of Discrete Breathers in a Triangular Lattice

Robert S. MacKay
Mathematics Institute, University of Warwick, UK

Vassilis Koukouloyannis
Theoretical Mechanics, Department of Physics,
Aristoteleion University of Thessaloniki, Greece

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Motivation

- We want to examine the possibility of existence of time non reversible breather-solutions in a network of time-reversible oscillators.
- In addition we want to determine the linear stability of these solutions

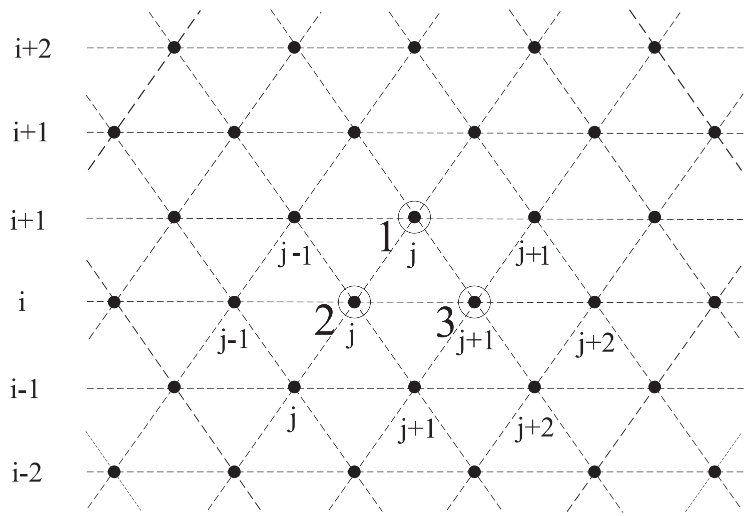
By the term time-reversible solution we mean that there exist initial conditions of the specific solution that satisfy

$$x_i(-t) = x_i(t) \quad \text{and} \quad p_i(-t) = -p_i(t) \qquad \forall \quad i = 1 \dots n$$

where n is the total number of degrees of freedom.

The System

The setting under consideration was proposed by MacKay & Aubry in 1994.



With on-site potential and nearest neighbor interaction.

$$H = H_0 + \varepsilon H_1 = \sum_{i,j=-\infty}^{\infty} \frac{p_{ij}^2}{2} + V_{ij}(x_{ij}) + \frac{\varepsilon}{2} \sum_{i,j=-\infty}^{\infty} \Big\{ (x_{ij} - x_{i-1,j})^2 + (x_{ij} - x_{i-1,j+1})^2 +$$
$$+ (x_{ij} - x_{i,j-1})^2 + (x_{ij} - x_{i,j+1})^2 + (x_{ij} - x_{i+1,j-1})^2 + (x_{ij} - x_{i+1,j})^2 \Big\}$$

The anticontinuous limit

In the uncoupled limit $\varepsilon = 0$ we consider the 3 encircled oscillators which we call "central" moving in identical T-periodic orbits with arbitrary phases, while the rest lie on the stable equilibrium. Consider the action-angle variables, where the action is defined by

$$I_i = \oint_{\gamma_i} p_j \mathrm{d} q_j$$

and w_i is the corresponding angle variable. The motion of the central oscillators is the described in these variables by

$$\begin{aligned} I_i &= \text{const.} \\ w_i &= \omega_i t + w_{i_0} \quad \text{mod} 2\pi \end{aligned}$$

This state describes a trivially localised and T-periodic motion for the full system which is denoted by z_0.

Since every periodic orbit is defined modulo a phase shift it is more consistent to work with the variables

$$\begin{aligned} \vartheta &= w_3 & A &= I_1 + I_2 + I_3 \\ \phi_1 &= w_1 - w_3 & J_1 &= I_1 \\ \phi_2 &= w_2 - w_3 & J_2 &= I_2 \end{aligned}$$

It is proven that the critical points of

$$H^{\text{eff}}(J_1, J_2, A, \phi_1, \phi_2) = \frac{1}{T} \oint H \circ z(t)\ \mathrm{d}t,$$

are in one-on-one correspondence to breather solutions. Where z is the periodic orbit which is continued by the previously described unperturbed one, with respect to constant A.

[1]. T. Ahn, R. S. MacKay and J-A. Sepulchre, Nonlinear Dynamics **25** *(2001),*

We cannot compute z, but, in the lowest order of approximation we can take z as the unperturbed one z_0 and get to first order with respect to ε

$$H^{\text{eff}} = H_0(J_1, J_2, A) + \varepsilon \langle H_1 \rangle_\vartheta (J_1, J_2, A, \phi_1, \phi_2),$$

where

$$\langle H_1 \rangle_\vartheta = \frac{1}{2\pi} \oint H_1 \circ z_0 \, \mathrm{d}\vartheta = \frac{1}{T} \oint H_1 \circ z_0 \, \mathrm{d}t = \langle H_1 \rangle_t .$$

So the condition of existence of breathers becomes, under non-degeneracy conditions

$$\frac{\partial \langle H_1 \rangle}{\partial \phi_i} = 0, \quad \det \left(\frac{\partial^2 \langle H_1 \rangle}{\partial \phi_i \partial \phi_j} \right) \neq 0$$

which coincides with the results of

[2]. V. Koukouloyannis and S. Ichtiaroglou, Phys. Rev. E **66** *(2002), 066602.*

A general triangular lattice

We consider a generic time-reversible oscillator by

$$x(t) = \sum_{n=0}^{\infty} A_n(I)\cos nw = \sum_{n=0}^{\infty} A_n(I)\cos n(\omega t + \vartheta).$$

Since $$H_1 = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_1x_3 - x_2x_3$$

$$\langle H_1\rangle = C(J) - \frac{1}{2}\left\{\sum_{n=1}^{\infty} A_{1n}A_{3n}\cos n\phi_1 + A_{2n}A_{3n}\cos n\phi_2 + A_{1n}A_{2n}\cos n\phi_3\right\}$$

$$\begin{aligned}
\frac{\partial\langle H_1\rangle}{\partial\phi_1} \quad &= \quad \frac{1}{2}\sum_{n=1}^{\infty} nA_n^2\left(\sin n\phi_1 - \sin n\phi_3\right) \quad &= \quad 0 \\
\frac{\partial\langle H_1\rangle}{\partial\phi_2} \quad &= \quad \frac{1}{2}\sum_{n=1}^{\infty} nA_n^2\left(\sin n\phi_2 + \sin n\phi_3\right) \quad &= \quad 0.
\end{aligned}$$

Considering $\phi_3 = \phi_2 - \phi_1$, this is satisfied $\forall A_n$ if $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\sin(n\phi_1) - \sin[n(\phi_2 - \phi_1)] \quad &= \quad 0 \\
\sin(n\phi_2) + \sin[n(\phi_2 - \phi_1)] \quad &= \quad 0.
\end{aligned}$$

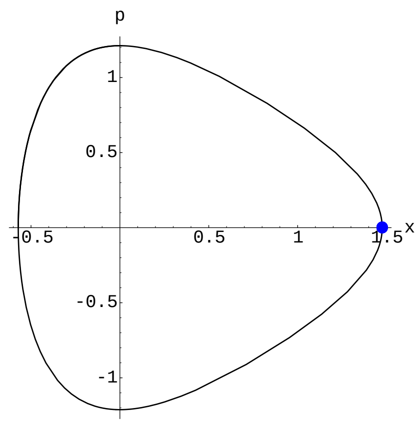
A general triangular lattice

This system has at least the solutions

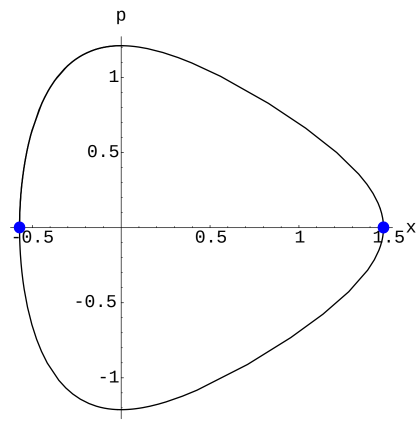
$$\phi_1 = 0 \quad \phi_2 = 0 \qquad 1^{st}\text{class}$$

$$\left.\begin{array}{ll} \phi_1 = 0 & \phi_2 = \pi \\ \phi_1 = \pi & \phi_2 = 0 \\ \phi_1 = \pi & \phi_2 = \pi \end{array}\right\} \quad 2^{nd}\text{class}$$

1^{st}class



2^{nd}class



which correspond to time-reversible breathers, and the solutions

$$\phi_1 = \frac{2\pi}{3} \quad \phi_2 = \frac{4\pi}{3}$$

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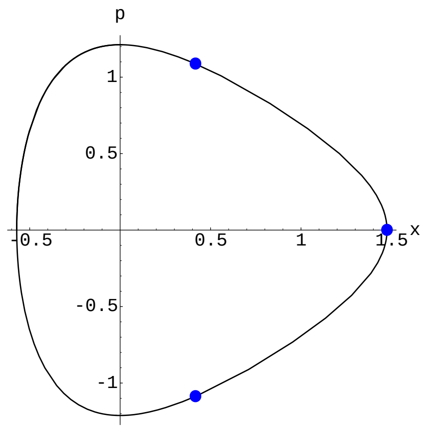


which correspond to time non–reversible breather solutions.

In [1]. it is also proven that the linear stability of the breather depends on the stability of the corresponding critical point of H^{eff}. So, we need the eigenvalues of the stability matrix

$$E = -\Omega D^2 H^{\text{eff}}$$

to lie in the imaginary axis for linear stability. By setting $H^{\text{eff}} = H_0 + \varepsilon\langle H_1\rangle$ this matrix becomes

$$E = \begin{pmatrix} -\varepsilon g_3 & -\varepsilon(g_1+g_2) & -\varepsilon(f_1+f_2) & \varepsilon f_3 \\ -\varepsilon(g_2-g_3) & \varepsilon g_3 & \varepsilon f_3 & -\varepsilon(f_2+f_3) \\ 2c+\varepsilon\dfrac{\partial^2\langle H_1\rangle}{\partial J_1^2} & c+\varepsilon\dfrac{\partial^2\langle H_1\rangle}{\partial J_1\partial J_2} & \varepsilon g_3 & \varepsilon(g_2-g_3) \\ c+\varepsilon\dfrac{\partial^2\langle H_1\rangle}{\partial J_1\partial J_2} & 2c+\varepsilon\dfrac{\partial^2\langle H_1\rangle}{\partial J_2^2} & \varepsilon(g_1+g_3) & \varepsilon g_3 \end{pmatrix}$$

$f_i = \frac{1}{2}\sum_{n=1}^{\infty} n^2 A_n^2 \cos n\phi_i,\ g_i = \frac{1}{2}\sum_{n=1}^{\infty} nA_n \frac{\partial A_n}{\partial I} \sin n\phi_i, \quad c = \frac{\mathrm{d}\omega}{\mathrm{d}J}$

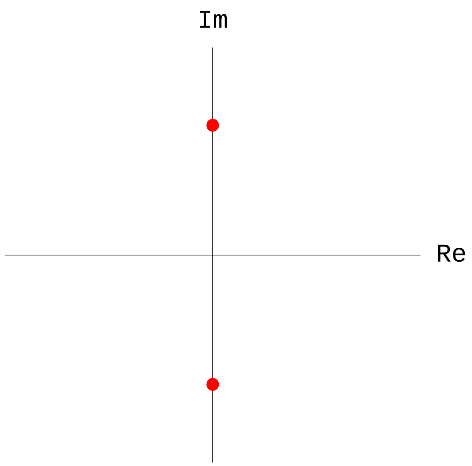
Linear stability of the first class of time-reversible solutions

The eigenvalues of the first class of the time-reversible solutions are

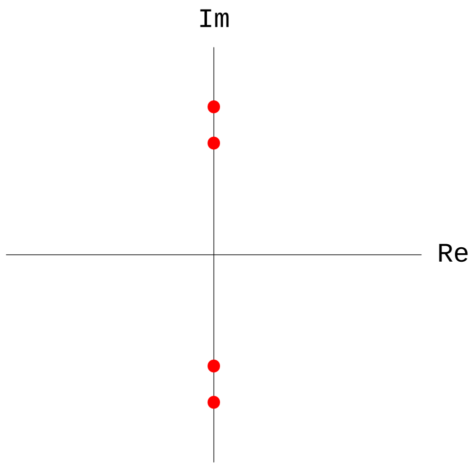
$$\lambda_{1,2,3,4} = \pm\sqrt{-3cf(0)}\sqrt{\varepsilon} + O(\varepsilon).$$

Since, $f(0) = \frac{1}{2}\sum_{n=1}^{\infty} n^2 A_n^2 > 0,$

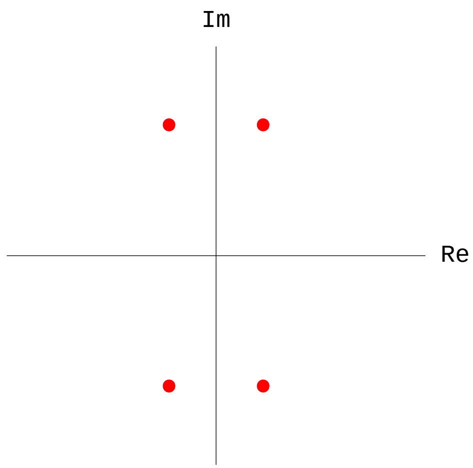
if $\varepsilon c > 0$, the leading order term implies linear stability. But it is of multiplicity 2 which could lead to instability due to higher order terms.



Multiple eigenvalue



Linearly stable scenario



Unstable scenario

The solution to this problem is the symplectic signature theory for fixed points of Hamiltonian systems, which states that if the quadrartic form which coresponds to

$$D^2 H^{\mathrm{eff}}$$

is definite, then the eigenvalues cannot leave the imaginary axis for any small perturbation. The quadratic form for this particular case is

$$\delta^2 H = \frac{3}{2}c\left(J_1 + J_2\right)^2 + \frac{1}{2}c\left(J_1 - J_2\right)^2 + \frac{1}{2}\varepsilon f\left(\phi_1 + \phi_2\right)^2 + \frac{3}{2}\varepsilon f\left(\phi_1 - \phi_2\right)^2$$

so it is definite if $\varepsilon c > 0$ and the breather is linearly stable.

Linear stability of the second class of time-reversible breather solutions

For the second class of reversible solutions we have

$$\lambda_{1,2} = \pm\sqrt{-c(2f(0)+f(\pi))}\sqrt{\varepsilon}+O(\varepsilon), \quad \lambda_{3,4} = \pm\sqrt{-3cf(\pi)}\sqrt{\varepsilon}+O(\varepsilon)$$

and since

$$2f(0)+f(\pi)>0$$

for this solution to be stable we need $f(\pi)$ & $\varepsilon c > 0$.
Since, in general in this case, the multiplicity of these eigenvalues is 1 there is no need for symplectic signature analysis.

Linear stability of time non-reversible case

The eigenvalues which correspond to the time non-reversible case are

$$\lambda_{1,2,3,4} = \pm i\sqrt{3cf(2\pi/3)}\sqrt{\varepsilon} + O(\varepsilon)$$

with corresponding quadratic form

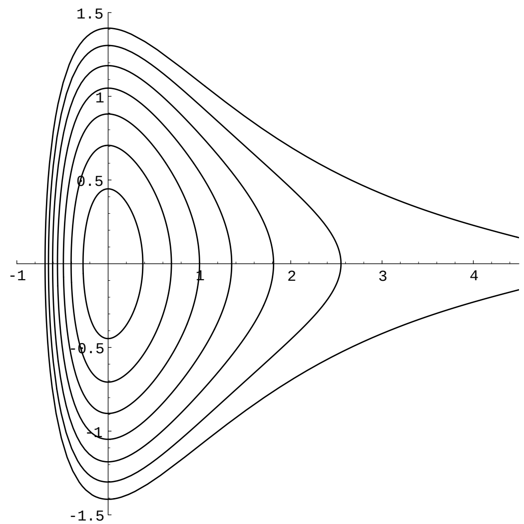
$$\begin{aligned} \delta^2 H &= 4c\left(2J_1 + J_2 + \frac{2\varepsilon g}{c^2}(\phi_1 - \phi_2)\right)^2 + \frac{3c}{2}\left(J_2 + \frac{\varepsilon g \phi_1}{c^2}\right)^2 \\ &\quad + \frac{\varepsilon}{2}\left(f - \frac{\varepsilon g^2}{c}\right)(\phi_1 + \phi_2)^2 + \frac{3\varepsilon}{2}\left(f - \frac{\varepsilon g^2}{c}\right)(\phi_1 - \phi_2)^2 \end{aligned}$$

which is definite if $\varepsilon f c > 0$

A lattice of coupled Morse oscillators

The Hamiltonian of the Morse oscillator is

$$H = \frac{p^2}{2} + \left(e^{-x} - 1\right)^2$$



while the averaged part of the perturbative term of the Hamiltonian of the full sysytem is

$$\langle H_1 \rangle = 2 \sum_{i=1}^{3} \int \arctan \frac{\sin \phi_i}{z - \cos \phi_i} \mathrm{d}\phi_i + C(J)$$

with $z = e^{2a}$ and $\cosh a = E^{-\frac{1}{2}}$.

The eigenvalues for the first class of time-reversible solutions are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{6}{z-1}}\sqrt{\varepsilon} + O(\varepsilon)$$

while for the second class they are

$$\lambda_{1,2} = \pm\sqrt{2\frac{z+3}{z^2-1}}\sqrt{\varepsilon} + O(\varepsilon),\ \ \lambda_{3,4} = \pm i\sqrt{\frac{6}{1+z}}\sqrt{\varepsilon} + O(\varepsilon).$$

This means that the time-reversible solutions are unstable.
On the other hand, the eigenvalues for the time non-reversible case are

$$\lambda_{1,2,3,4} = \pm i\sqrt{3\frac{2+z}{1+z+z^2}}\sqrt{\varepsilon} + O(\varepsilon),$$

which means, that these solutions are linearly stable.

Conclusions

- We prooved the existence of time-reversible as well as non-reversible discrete breathers in a network of time-reversible oscillators.
- We provide a systematic way of acquiring the linear stability of these breather-solutions.
- We applied these results in a triangular lattice of coupled Morse oscillators and realize that in this specific case the linearly stable breathers are the non-reversible ones.
- These results can be immediately generalised since we did not make use of any special symmetry property of the specific lattice.